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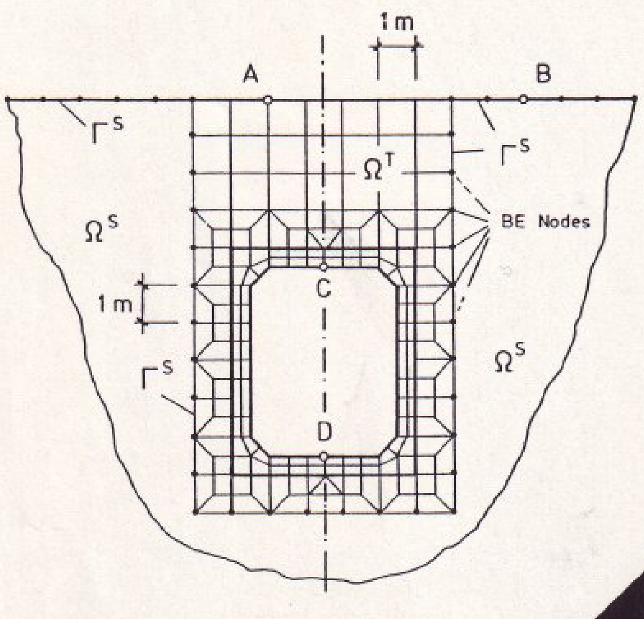

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Dynamic boundary integral "equation" method for unilateral contact problems

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The aim of the present paper is the extension of the method of boundary integral equations (B.I.E.) to dynamic unilateral contact problems. Using semidiscretization, with respect to time, and then the inequality constrained principle of minimum potential and the equivalent variational inequality formulation, we derive saddle point formulations for the problems using appropriate Langrangian functions. An elimination technique gives rise to a minimum 'principle' on the boundary with respect to the unknown normal displacements of the contact region, which has as parameters the velocities etc. of the previous time steps. It is also shown that the minimum problem is equivalent to a multivalued boundary integral equations problem involving symmetric operators. The theory is illustrated by numerical examples, which also treat the case of impact of the structure with its support. In order to achieve this last task, an appropriate time discretization scheme has been chosen. Numerical examples dealing with the seismic behaviour of two-dimensional structures supported by the ground are presented to illustrate the method.

Key Words: boundary elements, dynamics, impact, soil-structure interaction, time domain, unilateral contact, variational inequality

1. INTRODUCTION

In recent years a large number of problems involving unilateral constraints have been studied[1-4]. For static problems, variational or hemivariational inequalities expressing the principle of virtual or complementary virtual work in its inequality form are employed. These variational inequalities give rise to global or local minimum problems for the potential or the complementary energy, which after discretization permit a numerical treatment by using nonlinear algorithms.

For dynamic problems the corresponding variational or heminariational inequalities do not lead to minimum problems, but after a time discretization. Thus, we have to solve an appropriate minimum problem within each time step.

Important among the unilateral or inequality problems, are the unilateral contact problems which arise when a deformable body is in 'ambiguous' contact with a rigid or a deformable support or with another body. The term 'ambiguous' means that we do not know *a priori* which parts of the body are in contact with the support or the other body; this fact renders the problem unilateral. Our attention is focused to the case of a rigid support since all other types of unilateral contact problem can be reduced to this first case. This is the famous problem posed by Signorini in 1933 and studied in the static case by Fichera[5]. Here we deal with the dynamic problem. For the numerical treatment of this problem we have to solve, within each time step, an inequality constrained quadratic programming (Q.P.) problem with respect to the displacements (minimum of potential energy) or with respect to the stresses (minimum of complementary energy). Several works have been published which present techniques for the static or the dynamic problem, solving directly or indirectly, the arising minimization problem[2,6-12]. However, in all these techniques the size of the structure to be analysed must be small, since the stiffness of the whole structure is considered, or major changes in existing general finite element programs are required. Even the active constraint strategy[10,11] has the same disadvantage due to the low degree of automatisation concerning the determination of the active constraints.

In order to diminish the number of unknowns several elimination techniques of the internal degrees of freedom have already applied to static unilateral contact problems[13-19]. The most delicate elimination technique has been proposed in Panagiotopoulos *et al.*[17-19] and is based on Lagrangian formulations which lead, for the static problem, to an inequality constrained minimization problem on the contact zone only, thereby drastically diminishing the number of unknowns. However, after the discretization, full matrices are obtained instead of banded ones. The derived minimum problems on the boundary are equivalent to multivalued boundary integral equations (B.I.Es).

In order to treat the corresponding dynamic problem, first a discretization with respect to time is performed and then, within each time step, an inequality constrained minimum problem on the boundary is formulated, after the elimination of the internal degrees of freedom. Note that the resulting time-difference Q.P. problem on the boundary is equivalent to time-difference multivalued B.I.Es. From this standpoint the proposed method for the treatment of the dynamic unilateral contact problem, can be seen as a first attempt towards the formulation of a B.I.Es. method for dynamic unilateral contact problems.

2. FORMULATION OF THE PROBLEM AS A SADDLE POINT PROBLEM, WITHIN EACH TIME STEP

We consider a three-dimensional linear elastic body. The method we present here is general and holds also for plates, beams, etc., i.e., for all structures for which a Lagrangian formulation of the equilibrium problem is possible.

Let Ω be a subset of the three-dimensional Euclidean space $\mathbb{R}^3$ with a boundary Γ. Ω is occupied by a linear elastic body in its underformed state and is referred to an orthogonal Cartesian coordinate system $Ox_1x_2x_3$. Γ is decomposed into three mutually disjointed parts Γ_U, Γ_F and Γ_S. On Γ_U (respectively Γ_F) the displacements (respectively the tractions) are given and on Γ_S the Signorini-Fichera boundary conditions hold. We denote also by $\eta = \{\eta_i\}$, the outward unit normal vector to Γ by $S = \{S_i\} = \{\sigma_{ij}\eta_j\}$, the traction vector on the boundary, where $\sigma = \{\sigma_{ij}\}$ is the stress tensor. Further, let S_N (respectively S_T) be the normal (respectively the tangential) component of S with respect to Γ and let also U_N and U_T be the corresponding components of the displacement vector U. We denote by $\varepsilon = \{\varepsilon_{ij}\}$ the strain tensor (assumption of small strains) and by $C = \{C_{ijhk}\}$ $\{i, j, h, k = 1, 2, 3\}$. Hooke's tensor of elasticity which has the symmetry property

$$C_{ijhk} = C_{jihk} = C_{khij} \tag{1}$$

and the ellipticity property

$$C_{ijhk} \cdot \varepsilon_{ij} \cdot \varepsilon_{hk} \geqslant c \cdot \varepsilon_{ij} \cdot \varepsilon_{hk} \tag{2}$$

$$\forall \varepsilon = \{\varepsilon_{ij}\} \in \mathbb{R}^3, \quad c \text{ const.} > 0.$$

We have on Γ_U

$$u_i = U_i,\ U_i = U_i(x, t) \text{ given} \tag{3}$$

and on Γ_F

$$S_i = F_i,\ F_i = F_i(x, t) \text{ given.} \tag{4}$$

The Signorini-Fichera boundary condition states that if $u_N < 0$, then $S_N = 0$ (no contact), while if $u_N = 0$ then $S_N \leqslant 0$ (no contact), while if $u_N = 0$ then $S_N \leqslant 0$ (contact), or equivalently

$$S_N \leqslant 0,\ u_N \leqslant 0,\ u_N S_N = 0 \text{ on } \Gamma_S \tag{5}$$

Relations (5) is now completed with the condition in the tangential direction

$$S_{Ti} = C_{Ti},\ C_{Ti} = C_{Ti}(x, t) \text{ on } \Gamma_S \tag{6}$$

where C_T is a given tangential force distribution. It is interesting to note that the case of more realistic unilateral contact laws, e.g. with a linearly deformable support, can be reduced to the law (5) by enlarging the body Ω by fictitious linear-elastic springs along Γ_s having appropriate spring constants.

The equations of motion read

$$\sigma_{ij,j} + f_i(t) = \rho\ddot{u}_i + c\dot{u}_i \text{ in } \Omega \times (O, T) \tag{7}$$

$$\varepsilon_{ij} = \varepsilon_{ij}(\mathrm{u}) = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \text{ in } \Omega \times (O, T) \tag{8}$$

$$\sigma_{ij} = C_{ijhk} \cdot \varepsilon_{hk} \text{ in } \Omega \times (O, T) \tag{9}$$

$$u_i = u_{io}(x) \text{ at } t = 0 \tag{10}$$

$$\dot{u}_i = u_{i1}(x) \text{ at } t = 0 \tag{10a}$$

where u_{io} (respectively u_{i1}) denotes the initial displacements (respectively velocities), (O, T) is the time interval in which the motion of the body is observed, $\mathbf{f} = \{f_i\}$ represents the volume force vector, a comma denotes spatial differentiation, $\ddot{u}_i$ is the acceleration vector and ρ is the mass density. Let us assume that the damping term is proportional to the velocity $\dot{u}_i$ and let us denote the damping coefficient by $c > 0$.

For the time discretization of the problem, the method of m-step linear difference operators is applied. At time instant $t^{(p)}$ we assume that

$$\sum_{r=0}^{q} a^{(r)}u^{(p-r)} = \delta t \sum_{r=0}^{q} b^{(r)}\dot{u}^{(p-r)} \tag{11a}$$

and

$$\sum_{r=0}^{q} \gamma^{(r)}u^{(p-r)} = \mu(\delta t)^2 \sum_{r=0}^{q} b^{(r)}\ddot{u}^{(p-r)},\ p \geqslant q,\ p > 1 \tag{11b}$$

The coefficients $a^{(r)}$, $b^{(r)}$, $\gamma^{(r)}$, and μ depend on the finite-difference scheme chosen. We assume that the step size δt remains constant. Moreover, in order to have an implicit integration scheme we take $b^{(o)}$ to be nonzero. We obtain the relations

$$a^{(o)}u^{(p)} - \delta t b^{(o)}\dot{u}^{(p)} = -\mathbf{U}\mathbf{a} + \delta t\dot{\mathbf{U}}\mathbf{b} \tag{11c}$$

and

$$\begin{aligned} \mu(\delta t)^2 b^{(o)}\ddot{u}^{(p)} = \gamma^{(o)}u^{(p)} &= \mathbf{U}\gamma - \mu(\delta t)^2\ddot{\mathbf{U}}\mathbf{b} \\ &= \gamma^{(o)}u^{(p)} + \bar{\mathbf{U}}, \end{aligned} \tag{11d}$$

where

$$\mathbf{a} = [a^{(1)}, \ldots, a^{(9)}]^T,\ \mathbf{b} = [b^{(1)}, \ldots, b^{(q)}]^T,$$
$$\boldsymbol{\gamma} = [\gamma^{(1)}, \ldots, \gamma^{(q)}]^T \text{ and } \mathbf{U} = [u^{(p-1)}, \ldots, u^{(p-q)}].$$

Accordingly we may write within the p-time interval $(t,^{(p)}t^{(p)} + \delta t)$, after applying the above time discretization to the equations of motion, that (we omit the index p)

$$\sigma_{ij,j} + \bar{f}_i(t) = -Au_i \quad A > 0 \text{ in } \Omega \times (t, t + \delta t)$$

where

$$\bar{f}_i = f_i - q_i$$

and where we consider from now on only the behaviour of the structure in the time interval $(t, t + \delta t)$.

Let V be the set

$$V=\{v\,|\,v=\{v_i\},\ v_i=U_i\ i=1,2,3\text{ on }\Gamma_v\}. \quad (12)$$

Then, the kinematically admissible set for the displacements within the time interval $(t, t+\delta t)$ is

$$K=\{v\,|\,v=\{v_i\},\ v_i\in V,\ v_N\leqslant 0\text{ on }\Gamma_s\} \quad (13)$$

We denote by $(\bar{f}, v)$ the work of the force $\bar{f}=\{\bar{f}_i\}$ for the displacement $v=\{v_i\}$ on $\Omega x\,(t, t+\delta t)$ and by $[\bar{f}, v]_{\Gamma}$ the corresponding work on $\bar{\Gamma}\subset\Gamma$ (i.e. $\int_\Omega \bar{f}_i v_i d_\Gamma$) etc. Further, let

$$(u,v)=A\int_\Omega u_i U_i d\Omega \quad A>0 \quad (14a)$$

and

$$a(u,v)=(C\varepsilon(u),\varepsilon(v))=\int_o C_{ijhk}\cdot\varepsilon_{ij}(u)\cdot\varepsilon_{hk}(v)d\Omega \quad (14b)$$

be the bilinear form of elasticity and let Π be the potential energy within the interval $(t, t+\delta t)$, i.e.,

$$\Pi(v)=\tfrac{1}{2}a(v,v)+\tfrac{1}{2}(v,v)-(\bar{f},v)-[C_T,v_T]_{\Gamma_s}-[F,v]_{\Gamma_r} \quad (15)$$

We know that[5]

$$\Pi(u)=\{\Pi(v)\,|\,v\in K\}\text{ in }(t,t+\delta t) \quad (16)$$

characterizes the position of equilibrium in each time interval. Problem (14) has one and only one solution (for $v_i\in H'(\Omega)$ – the Sobolev-space- $C_{ijhk}\in L^\infty \bar{f}_i$, C_{Ti}, $F_i\in L^2$, $U_i\in H^{1/2}$) which satisfies equivalently the variational inequality

$$u\in K,\ a(u,v-u)+(u,v-u)-l(v-u)\geqslant 0\ \forall v\in K \quad (17)$$

with $l(v)=(\bar{f},v)+[F,v]_{\Gamma_r}+[C_T,v_T]_{\Gamma_s}$. Relations (16) or (17) are the primal formulations of the boundary value problem within the time interval δt.

For the mixed formulation let us introduce the convex subset (which is closed in the previously mentioned functional framework)

$$L=\{\mu_N\,|\,\mu_N\leqslant 0\text{ on }\Gamma_s\}. \quad (18)$$

We perform the translation

$$\bar{u}=u-u_0,\ \bar{v}=v-u_0 \quad (19)$$

where u_o, is a kinematically admissible displacement field, i.e. such that $u_{oi}=U_i$ on Γ_v and

$$\bar{u},\bar{v}\in V_o=\{v\,|\,v=\{v_i\},\ v_i\in\bar{V},\ v_i=0\text{ on }\Gamma_v\}. \quad (20)$$

Then (17) takes the form: Find $u=\bar{u}+u_o\in K$ such that

$$a(\bar{u},\bar{v}-\bar{u})+(\bar{u},\bar{v}-\bar{u})-l(\bar{v}-\bar{u})$$
$$+a(u_o,\bar{v}-\bar{u})+(u_o,\bar{v}-\bar{u})\geqslant 0 \quad (21)$$
$$\forall v=\bar{v}+u_o\in K$$

and (16) becomes

$$\bar{\Pi}(\bar{u})=\min\{\bar{\Pi}(\bar{v})\,|\,\bar{v}\in\bar{K}\}, \quad (22)$$

where

$$\bar{\Pi}(\bar{v})=\tfrac{1}{2}a(\bar{v},\bar{v})+\tfrac{1}{2}(\bar{v},v)-l(\bar{v})+a(u_o,\bar{v})=\Pi(v)-\Pi(u_o) \quad (23)$$

and

$$\bar{K}=\{\bar{v}\,|\,\bar{v}_N+u_{ON}\leqslant 0\text{ on }\Gamma_s\}. \quad (24)$$

Note that u_o may represent any other initial strain field (temperature distributions, given dislocations etc.) We denote by $\bar{l}$ the functional

$$\bar{l}(\bar{v})=l(\bar{v})-a(u_o,\bar{v})-(u_o,\bar{v}) \quad (25)$$

Here μ_N is the Lagrange multiplier for the problem. Through this Lagrange multiplier we introduce the boundary condition (5) on Γ_s.

We have that

$$\inf_{\bar{K}}\{\bar{\Pi}(\bar{v})\}=\inf_{V_o}(\bar{\Pi}(\bar{v})+I_{\bar{K}}(\bar{v}))$$

where

$$I_{K(\bar{v})}=\begin{cases}0 & \text{in } \bar{v}+u_{ON}\text{ on }\Gamma_s\\ \infty & \text{otherwise}\end{cases} \quad (26)$$

But

$$I_{\bar{K}}(\bar{v})=\sup_{\mu_N\leqslant 0}\,[-[\mu_N,\bar{v}_N+u_{ON}]_{\Gamma_s}] \quad (27)$$

and therefore

$$\inf_{\bar{K}}\bar{\Pi}(\bar{v})=\inf_{\bar{v}\in V_o}\sup_{\mu_N\in L}\{\bar{\Pi}(\bar{v})-[\mu_N,\bar{v}_N+u_{ON}]_{\Gamma_s}\}. \quad (28)$$

Thus the Lagrangian of the problem is a real-valued function $\mathscr{L}$ on $V_o\times L$ defined within the time interval $(t, t+\delta t)$ by the relation

$$\mathscr{L}(\bar{v},\mu_N)=\tfrac{1}{2}a(\bar{v},\bar{v})+\tfrac{1}{2}(\bar{v},\bar{v})-[\mu_N,\bar{v}_N+u_{ON}]_{\Gamma_s}$$
$$-[C_T,\bar{v}_T]_{\Gamma_s}-[F,\bar{v}]_{\Gamma_r}-(\bar{f},\bar{v})$$
$$+a(u_o,\bar{v})+(u_o,\bar{v}). \quad (29)$$

Thus, the mixed variational formulation of the semi-discretized Signorini-Fichera problem now reads: Find the saddle point

$$\bar{w},\lambda_N\in V_o\times L\text{ of }\mathscr{L}\text{ on }V_o\times L,\text{ i.e.}$$
$$\mathscr{L}(\bar{W},\mu_N)\leqslant\mathscr{L}(\bar{W},\lambda_N)\leqslant\mathscr{L}(\bar{v},\lambda_N)\forall\bar{v}\in V_o,$$
$$\mu_N\in L\text{ in }(t,t,+\delta t) \quad (30)$$

In the previously mentioned functional framework we can prove by the methods given in Ekeland and Temam[20] (cf. also Refs 21, 22) that problem (30) admits a unique solution $\{\bar{W},\lambda_N\},\in V_o\times L$ within the time interval $(t, t, +\delta t)$ such that $\bar{W}=\bar{u}\in\bar{K}$ and $\lambda_N=S_N(\bar{u})$ on Γ_s.

3. FORMULATION WITHIN EACH TIME STEP WITH RESPECT TO THE DISPLACEMENTS OF THE CONTACT AREA

Let us consider again the saddle point formulation (30) which is now written in the form: Find $\{\bar{w},\lambda_N\}\in V_o\times L$ such as to satisfy the problem

$$\mathscr{L}(\bar{w},\lambda_N)=\inf_{V_o}\sup_{L}\mathscr{L}(\bar{v},\mu_N)=\sup_{L}\inf_{V_o}\mathscr{L}(\bar{v},\mu_N)$$
$$\text{in}(t,t+\delta t) \quad (31)$$

We set

$$\inf_{V_o}\mathscr{L}(\bar{v},\mu_N)=\bar{\Pi}_1(\mu_N) \quad (32)$$

assuming for the moment that $\mu_N \in L$ is given. Then (32) is equivalent to the following bilateral problem: Find $\tilde{u} = \tilde{u}(\mu_N) \in V_o$ such that

$$a(\tilde{u}, \bar{v}) + (\tilde{u}, \bar{v}) - [\mu_N, \bar{v}_N]_{\Gamma_s} - (\bar{f}, \bar{v}) - [F, \bar{v}]_{\Gamma_F} - [C_T, \bar{v}_T]_{\Gamma_s} + a(u_o, \bar{v}) + (u_o, \bar{v}) = 0 \ \forall \bar{v} \in V_o. \qquad (33)$$

Relation (33) is the expression of the principle of virtual work for a fictive structure resulting from the initial unilateral one, by eliminating the unilateral constraints on Γ_s and by adding the corresponding reactions μ_N. The position of equilibrium of this structure is characterized within the interval $(t, t+\delta t)$ by the minimization problem (32) for the potential energy of this fictive structure. The solution $\tilde{u}$ of (33) can be considered, due to the linearity of (33), as the sum of $\tilde{u}_1 \in V_o$ and $\tilde{u}_2 \in V_o$ which are solutions of the two following bilateral problems:

$$a(\tilde{u}_1, \bar{v}) + (\tilde{u}_1, \bar{v}) - (\bar{f}, \bar{v}) - [F, \bar{v}]_{\Gamma_F} - [C_T, \bar{v}_T]_{\Gamma_s} + a(u_o, \bar{v}) + (u_o, \bar{v}) = 0 \ \forall \bar{v} \in V_o \qquad (34)$$

$$a(\tilde{u}_2, \bar{v}) + (\tilde{u}_2, \bar{v}) - [\mu_N, \bar{v}_N]_{\Gamma_s} = 0 \ \forall \bar{v} \in V_o \qquad (35)$$

respectively. Obviously, both problems describe the equilibrium configuration within each time step of two bilateral structures resulting from the initial one, by considering at each point springs with constants $A > 0$, by ignoring the unilateral support, and by assuming that the appropriate boundary parts have zero loading.

In the case of (34), the structure is loaded by the forces f in Ω and C_T on Γ_s tangentially and F on Γ_F, whereas on Γ_s the normal loading is zero. Moreover, the initial displacement field u_o is taken into account. In the case of (35), the structure is subjected to normal forces μ_N on Γ_s and we assume zero forces in Ω, on Γ_F, and in the tangential direction on Γ_s. Accordingly, the solutions $\tilde{u}_1$ and $\tilde{u}_2$ are uniquely determined, as it is well known from the classical (bilateral) elasticity[23]. For the arising two bilateral structures we can write the solution in terms of Green's operator G. This operator is the 'same' for the two structures due to the 'same' type of boundary conditions holding in both cases (see Fig. 1). Thus, we may write in both cases the solution of the problem as follows:

$$\tilde{u}_1 = G(\bar{l}),\ \tilde{u}_2 = G(\mu_N),\ \tilde{u} = \tilde{u}_1 + \tilde{u}_2,\ \bar{e} = \{\bar{f}, F, C_T, u_o\}, \quad \text{in } (t, t+t\delta). \qquad (36)$$

Note that the as yet, unknown force distribution μ_N must be admissible in the sence of (18), i.e. $\mu_N \in L$. Thus from (32), (34) and (35) we obtain by setting $\bar{v} = \tilde{u}_1$ in (34) and $\bar{v} = \tilde{u}_2$ in (35) that

$$\begin{aligned}\tilde{\Pi}_1(\mu_N) &= \mathscr{L}(\tilde{u}, \mu_N) = \tfrac{1}{2}\tilde{a}(\tilde{u}, \tilde{u}) - [\mu_N, \tilde{u}_N + u_{ON}]_{\Gamma_s} \\ &\quad - [C_T, \tilde{u}_T]_{\Gamma_s} - (\bar{f}, \tilde{u}) - [F, \tilde{u}]_{\Gamma_F} + \tilde{a}(u_o, \tilde{u}) = \\ &\quad -[\mu_N, \tilde{u}_{N_1}]_{\Gamma_s} - \tfrac{1}{2}[\mu_N, \tilde{u}_{N_2}]_{\Gamma_s} - \tfrac{1}{2}(\bar{f}, \tilde{u}_1) \\ &\quad - \tfrac{1}{2}[F, \tilde{u}_1]_{\Gamma_F} - \tfrac{1}{2}[C_T, \tilde{u}_{T_1}]_{\Gamma_s} - [\mu_N, u_{ON}]_{\Gamma_s} \\ &\quad + \tfrac{1}{2}\tilde{a}(u_o, \tilde{u}_1) = \\ &\quad -[\mu_N, [G(\bar{l})]_N]_{\Gamma_s} - \tfrac{1}{2}[\mu_N, [G(\mu_N)]_N]_{\Gamma_s} \\ &\quad + \tfrac{1}{2}\tilde{a}(u_o, G(\bar{l})) - \tfrac{1}{2}(\bar{f}, G(\bar{l})) - \tfrac{1}{2}[F, G(\bar{l})]_{\Gamma_F} \\ &\quad - \tfrac{1}{2}[C_T, G(\bar{l})]_{\Gamma_s} - [\mu_N, u_{ON}]_{\Gamma_s}\end{aligned} \qquad (37)$$

where

$$\tilde{a}(u, v) = a(u, v) + (u, v) \qquad (38)$$

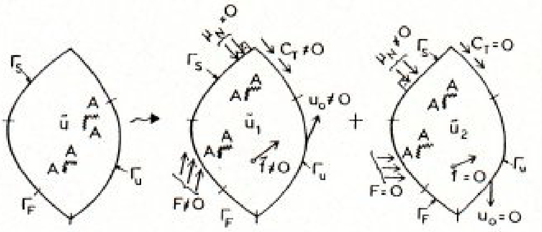

Fig. 1. The problem decomposition (force method) within each time step

Further, we denote by $\tilde{b}$ the bilinear form

$$\tilde{b}(\mu_N, v_N) = [\mu_N, [G(v_N)]_N]_{\Gamma_s} \qquad (39)$$

and by $\tilde{\gamma}$ the linear form

$$\tilde{\gamma}(\mu_N) = -[\mu_N, [G(\bar{l})]_N]_{\Gamma_s} - [\mu_N, u_{ON}]_{\Gamma_s} \qquad (40)$$

Thus,

$$\tilde{\Pi}_1(\mu_N) = -\tfrac{1}{2}\tilde{b}(\mu_N, \mu_N) + \tilde{\gamma}(\mu_N) - \tfrac{1}{2}(\bar{f}, G(\bar{l})) - \tfrac{1}{2}[F, G(\bar{l})]_{\Gamma_F} - \tfrac{1}{2}[C_T, G(\bar{l})]_{\Gamma_s} + \tfrac{1}{2}a(u_o, G(\bar{l})) + \tfrac{1}{2}(u_o, G(\bar{l})). \qquad (41)$$

From equations (31), (32) and (41) we can obtain the following minimization problem with respect to the unknown boundary tractions μ_N:

$$\min\{\Pi_1(\mu_N) = \tfrac{1}{2}\tilde{b}(\mu_N, \mu_N) - \tilde{\gamma}(\mu_N) \mid \mu_N \in L\}. \qquad (42)$$

The term $-\frac{1}{2}(\bar{f}, G(\bar{l})) - \frac{1}{2}[F, G(\bar{l})]_{\Gamma_F} - \frac{1}{2}[C_T, G(\bar{l})]_{\Gamma_s} + \frac{1}{2}\tilde{a}(u_o, G(l))$ does not depend on μ_N and therefore it can be omitted. We denote obviously by λ_N the solution of this problem. Analogously to (17) we can show now that $\lambda_N = S_N(\bar{u})$ on Γ_s, where $\bar{u} \in \tilde{K}$ is the solution of the problem (22). Moreover, in the functional framework introduced previously for the data of the problem in the interval $(t, t+\delta t)$, we can show that problem (42) has a unique solution[19]. Note that the symmetry of $\tilde{b}(.,.)$ results from Betti's theorem[17].

It is worth noting that (42) is equivalent to the variational inequality:

Find $\lambda_N \in L$ such as to satisfy

$$\tilde{b}(\lambda_N, \mu_N - \lambda_N) - \tilde{\gamma}(\mu_N - \lambda_N) \geqslant 0 \ \forall \mu_N \in L \qquad (43)$$

and to the 'multivalued' integral equation on the boundary part Γ_s of the structure which reads

$$\gamma - \tfrac{1}{2}\,\mathrm{grad}\; b(\lambda_N, \lambda_N) \in \partial I_L(\lambda_N) \qquad (44)$$

where $I_L(\lambda_N)$ is the indicator of the admissible se L i.e.

$$I_L(\lambda_N) = \begin{cases} 0 & \text{if } \lambda_N \leqslant 0 \quad (\text{i.e. } \lambda_N \in \mathrm{L}) \\ \infty & \text{otherwise} \quad (\text{i.e. } \lambda_N \notin \mathrm{L}). \end{cases} \qquad (45)$$

Relation (43) is equivalent to (42) by some well-known results of the theory of variational inequalities[1,2]. Relation (44) is equivalent to (43) by the definition of the subdifferential as shown in Ref. 20.

From the mechanical meaning of (43) we obtain an easy method for the calculation of corresponding discrete form. We refer in this context to Refs 17 and 18, with the additional remark that the work of the fictitious springs with constant A must be taken into account.

Now we apply to (45) the duality theory for variational inequalities[24] and we obtain the following minimum problem with respect to the unknown boundary normal displacements v_N:

$$\min\{\Pi_2(v_n) = \tfrac{1}{2}\tilde{\delta}(v_n, v_n) - \tilde{J}(v_n)/v_n \leqslant 0\} \tag{46}$$

where $\tilde{\delta}$ and $\tilde{J}$ are obtained from $\bar{b}$ and $\tilde{\gamma}$ respectively, by applying the relations (3.6) and (3.7) of Ref. 24 which are based on duality theory of convex functionals. Note that δ is a symmetric bilinear form which is coercive in the present problem. Also, the physical meaning of this duality implies an easy method for the calculation of $\tilde{\delta}$ and $\tilde{J}$ as we shall see in the next section. We notice here two equivalent formulations of (46). The first is: Find $u_N \in K$ (cf. (13)) such as to satisfy the variational inequality

$$\tilde{\delta}(u_N, v_N - u_N) - \tilde{J}(v_N - u_N) \geqslant 0 \ \forall v_N \in K \tag{47}$$

The second is: Find $u_N \in K$ solution of the multivalued B.I.E.

$$\tilde{J} - \tfrac{1}{2}\text{grad } \tilde{\delta}(u_N, u_N) \in \partial I_K(u_N). \tag{48}$$

4. NUMERICAL TREATMENT AND EXAMPLES

Out of the existing time integration schemes only direct integration methods can be used, due to the unilateral constraints which make the problem non-linear. The solution will be obtained using implicit and unconditionally stable time integration algorithms. Explicit and conditionally stable algorithms are not applicable, because the time step for them is to be chosen on the basis of formulas containing the frequencies of the system and these formulas do not apply to the present problem, since no frequencies can be defined in inequality problems in the usual sense[2].

For the numerical solution of the problem, the weighted residual method proposed by Zienckiewicz, Wood and Taylor[25], has been used. The method interpolates independently the displacement and velocity vectors and does not require computation of acceleration terms. This is a significant advantage for the present problem, since the calculation of 'initial' accelerations just after impact, would require additional computational effort and also additional computer storage for also storing the triangularised (consistent mass) matrix M.

Within each time step the discrete forms of $\tilde{\delta}$ and $\tilde{J}$ may be calculated in the following way: We consider first a structure Ω_o' obtained from the initial one by assuming $U_i = 0$ on Γ_v, $T_T = 0$ on Γ_s and $T_i = 0$ on Γ_s. The stiffness of this structure is appropriately modified by the frictious springs introduced by the time discretization scheme. Then we analyse this structure by imposing a unit normal displacement on the first node of Γ_s and zero normal displacements on the other nodes of Γ_s. The solution of this kinematically overconstrained structure Ω_o' gives the corresponding normal reactions of all the m-nodes of Γ_s. They constitute the first column of a matrix D. We repeat this procedure for the second node etc. and thus we form the whole symmetric matrix $\tilde{D}$. For each time step the normal reactions of the nodes of Γ_s for a structure Ω_o'' having U_i displacements on Γ_v, $T_T = C_T$ on Γ_s, $T_i = F_i$ on Γ_F, and loading $\tilde{J}$ inside Ω_0'', constitute a vector $\tilde{z}$. Then the solution of the discrete $Q.P.$ problem:

$$\min\{\tfrac{1}{2}v^T\tilde{D}v - \tilde{z}^T v \,|\, v \leqslant 0\} \tag{49}$$

where $v = \{v_{N1}, \ \ldots \ v_{Nm}\}$ gives the unknown normal displacements on Γ_s within the time step considered. Note that if for the calculation of D and $\tilde{z}$ the classical B.I.E.M. is used then we obtain a nonsymmetric matrix $\tilde{D}$[26].

As a first example, the seismic behaviour of a two dimensional structure supported by the ground is studied. The geometry and data of the structure are indicated in Fig. 2.

The ground is considered to be very stiff (e.g. rock) and the motion can be applied directly to the soil-structure interface. This motion is considered to be the horizontal harmonic excitation shown in Fig. 3.

The boundary conditions are the following: the soil-structure interface 33-40 (base), is firmly fixed, i.e. bilateral boundary conditions hold, while a part of the interface 1–12 has unilateral contact conditions with the ground.

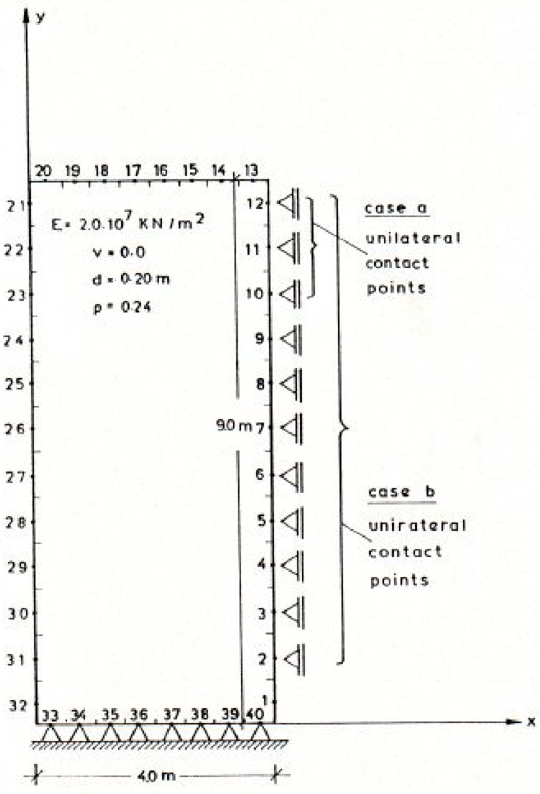

Fig. 2. The geometry and data of the examined problem

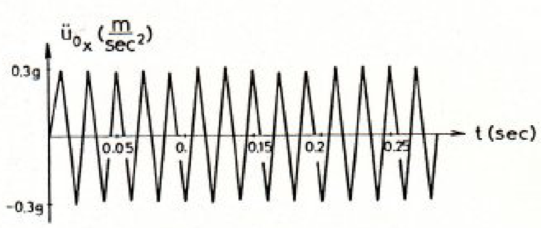

Fig. 3. The ground acceleration

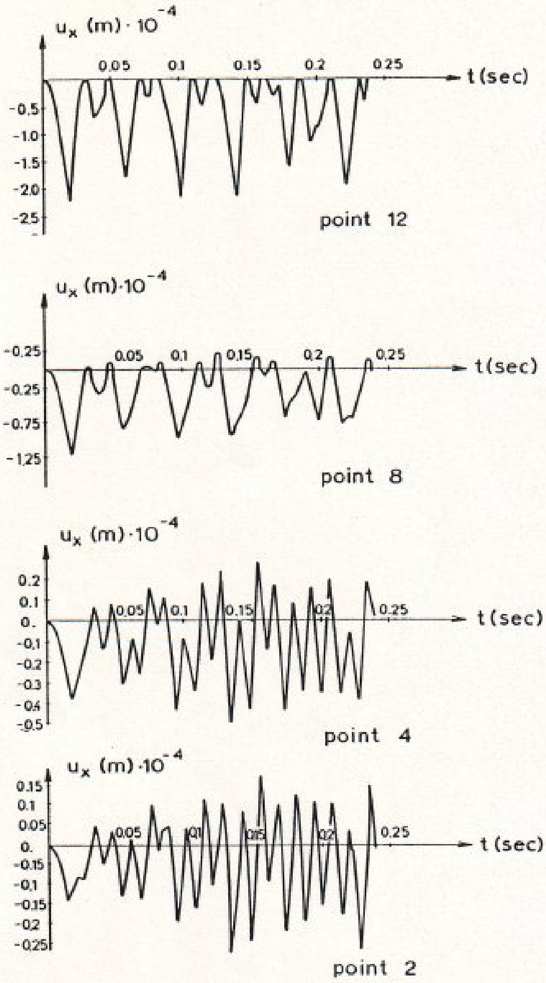

Fig. 4. Oscillations (x-displacements) of certain points of the wall for 3 points of possible contact (s. Fig. 2)

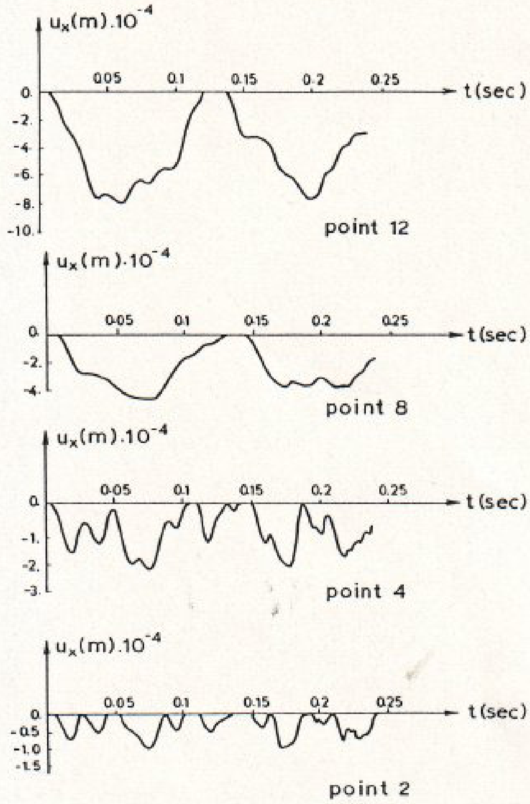

Fig. 5. Oscillations (x-displacements) of certain points of the wall for 11 points of possible contact (s. Fig. 2)

The vibration of the wall is considered undamped in the sense that C (cf equation (7)) is taken to be zero. When evaluating the dynamic response of the structure it is necessary to consider the collisions with the ground, at the interface 1–12, as the wall wobbles back and forth. The horizontal velocity of a point i before contact is $\dot{u}_i^-$. When the point gets into contact with the ground a part of the kinetic energy is lost. It is reasonable to postulate a perfectly inelastic collision which dissipates the whole kinetic energy. Thus, the velocity of the point i just after impact ($\dot{u}_i^+$) is equal to zero. Accordingly, although C was taken equal to zero, an amount of damping is considered due to perfectly plastic collision.

For the time discretization of the equations of motion, the algorithm of Zienckiewicz, Wood and Taylor[25] with a $\varphi = 0.5$ is used. For the numerical solution of the strictly convex quadratic programming problem (49), a modification of Hildreth D' Esopo's iterative method is used[12]. At each time step the convergence of the algorithm is very rapid. When a node i contacts the ground, the

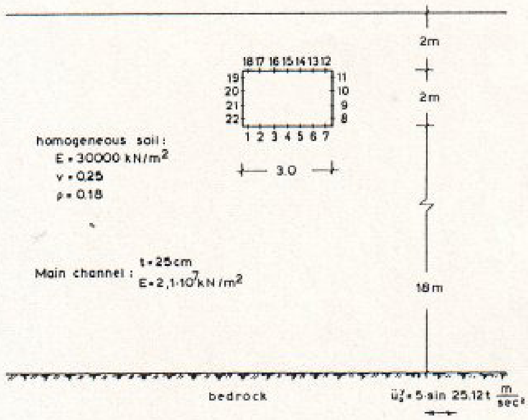

Fig. 6. The geometry and data of the problem

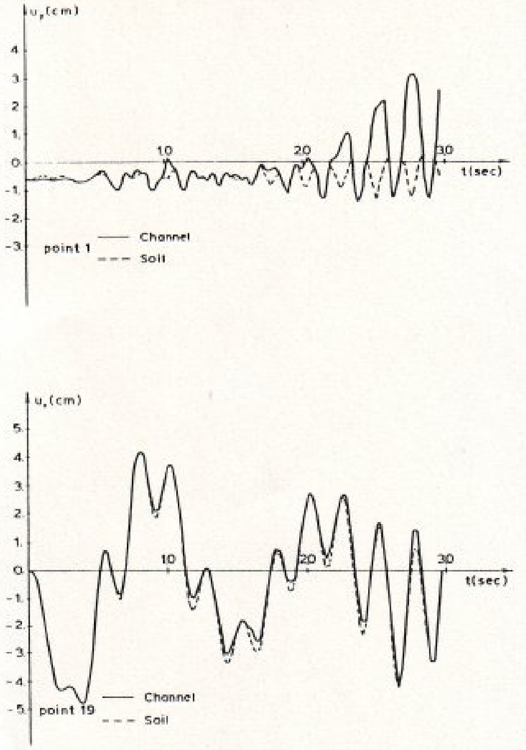

Fig. 7. Oscillations of certain channel-soil points for unilateral contact conditions

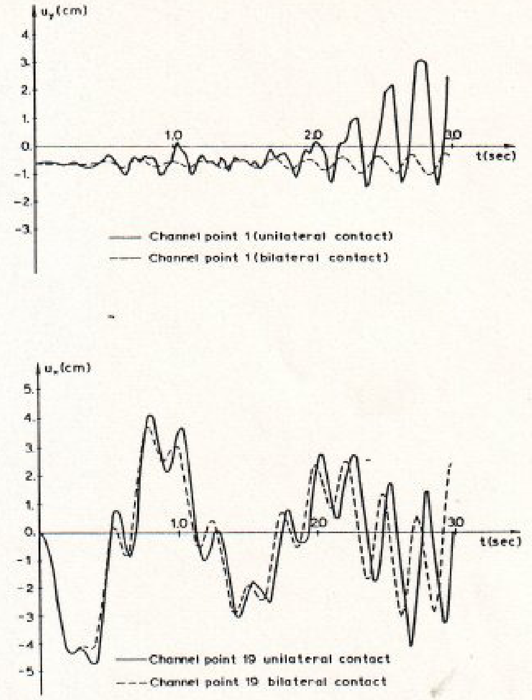

Fig. 9. Oscillation of points 1 and 19 of the Channel for unilateral and bilateral soil-structure conditions

condition $\dot{u}_i^+ = 0$ is imposed as 'initial' condition for the next time step. The computer code constructed is a general one and can take into account any other type of velocity changes due to impact (e.g. velocity reversal in the case of elastic impact etc).

In Fig. 4 the response of the system with three unilateral constraints is presented. Some of the examined points are constrained and some other not. In Fig. 5 the system with eleven unilateral constraints is presented. As it is expected the response of the wall for the two different support conditions is extremely different.

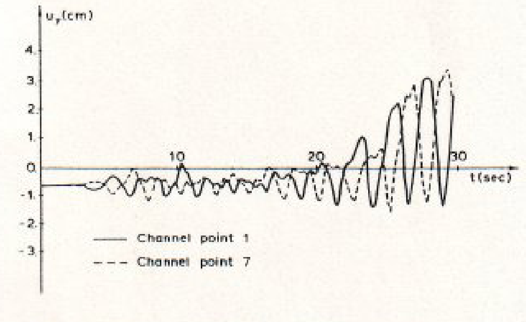

Fig. 8. y-y displacements of points 1 and 7

The computer code can effectively treat hundreds of unilateral constraints; actually the same problem size which treats the classical direct boundary integral equation code. The present examples have been calculated at a 386 PC and about 5 minutes are needed for 400 time steps.

As a second example we examine the problem of Fig. 6. It is a main channel buried in a linear elastic homogeneous soil, supported by a rigid bedrock on which a seismic excitation acts. A sinusoidal strong acceleration is given at the bedrock. Unilateral contact conditions with nonprevented sliding are taken into account between the channel and the soil. In Fig. 7 the time history of the displacements of certain points of the channel in relation with the same points of the soil, are presented. In Fig. 8 the displacements in the *y-y* direction of the two corners of the channel are shown. Finally in Fig. 9 displacements of the channel taken by a bilateral and also by a unilateral contact channel-soil assumption are compared.

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