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A boundary integral equation approach to elastodynamic inequality problems and applications

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In the present paper the method of boundary integral equations (BIE) is extended to dynamic inequality problems involving convex energy superpotentials, i.e. to problems involving monotone, possibly multivalued, relations between reactions and displacements, stress and strains, etc. Using semidiscretization with respect to time the authors obtain, within each time step, a minimum potential energy formulation, the equivalent variational inequality formulation and some equivalent saddle point formulations using appropriate Lagrangian functions. An elimination technique gives rise to minimum 'principles', on the boundary with respect to the unknown displacements or stresses of the time step under consideration; parameters are the velocities, etc., of the previous time step. It is also shown that these minimum problems are equivalent to multivalued boundary integral equation problems. The theory is illustrated by numerical examples, which also treat the case of impact of the structure with the support.

Key words: boundary integral method, variational inequality problems.

1 INTRODUCTION

In recent years several problems, for which the principles of virtual and complementary virtual work of power hold in inequality form, have been studied (see, e.g. Refs 4, 7, 8, 12, 13, 14). The inequality expressions of these principles can be classified into two types for static problems: the variational and the hemivariational inequalities. Then the derived inequality expressions lead to global or local minimum problems for the potential, or for the complementary energy, which after discretization permit the numerical treatment by means of nonlinear optimization algorithms.

For dynamic problems the corresponding variational or hemivariational inequalities may be formulated as minimum problems, only after an appropriate discretization. Thus, within each time step, results a minimum problem.

Among the unilateral or inequality problems an important class are the unilateral contact problems which arise when a deformable body is in 'ambiguous' contact with a rigid or a deformable support, or with another body, with or without friction (see, e.g. Refs 1, 3, 9, 10, 11, 16). The term 'ambiguous' means that it is not known *a priori* which parts of the body are in contact with the support or with the other body; moreover it is not known *a priori* which parts of those being in contact have an adhesive or a sliding friction.[18] Another class of problems are those of plastic hinges in plates; here the extent of the plastic hinge along the boundary is not known *a priori*. It has been shown, see Ref. 14, that all inequality problems involving monotone, possibly multivalued, stress-strain or reaction-displacement relations, or equivalently convex 'superpotentials', lead to variational inequalities. Thus we have to solve within each time step an inequality constrained convex programming problem with respect to the displacements (minimum of the potential energy), or with respect to the stresses (minimum of complementary energy).

However, until now the size of the structure considered should be small, due to the large number of

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unknowns. In order to diminish the number of unknowns an elimination technique of the internal degrees of freedom has been proposed by Panagiotopoulos[15] for static variational inequalities. This technique is extended here to dynamic variational inequalities: within each time step a parametric Lagrangian formulation of the initial variational inequality leads to time parametric minimum problems on the boundary, which are equivalent to multivalued boundary integral equations (BIEs) within each time step, i.e. producing time-difference multivalued BIEs. Inclusion of partial or total velocity reversal into this model permits the rational consideration of impact shocks. Numerical examples from aseismic design and from dynamic plasticity illustrate the theory.

This paper does not deal with the corresponding hemivariational inequality problems (see Refs 12, 13 and 14), where the lack of convexity and the resulting nonuniqueness of the solution within each time step have left, until now, many unsolved problems.

2 THE DYNAMIC VARIATIONAL INEQUALITY AS A SADDLE POINT PROBLEM, WITHIN EACH TIME STEP

The authors consider a three-dimensional linear elastic body; the method presented is general and holds also for plates, beams, etc., i.e. for all structures permitting a Langrangian formulation of the equilibrium problem.

Let Ω be a subset of the three-dimensional Euclidean space R^3 with a boundary Γ. Ω is occupied by a linear elastic body in its undeformed case which is referred to an orthogonal cartesian coordinate system $Ox_1x_2x_3$. Γ is decomposed into three mutually disjoint parts Γ_U, Γ_F and Γ_S. On Γ_U (with respect to Γ_F) the displacements (with the respect to the tractions) are given, and on Γ_S boundary conditions giving rise to variational inequalities hold. Let us denote by $\mathbf{n} = \{n_i\}$ the outward unit normal vector to Γ and by $\mathbf{S} = \{S_i\} = \{\sigma_{ij}, \cdot n_j\}$ the traction vector on the boundary, where $\sigma = \{\sigma_{ij}\}$ is the stress vector. Further let S_N (with respect to S_T) be the normal (with respect to the tangential) component of S with respect to Γ; let also u_N and u_T be the corresponding components of the displacement u. The authors denote the strain tensor by $\epsilon = \{\epsilon_{ij}\}$ (assumption of small strains), and by $C = \{C_{ijhk}\}\{i, j, h, k\} = \{1, 2, 3\}$ Hooke's tensor of elasticity which has the symmetry property:

$$C_{ijhk} = C_{jihk} = C_{khij} \tag{1}$$

and the ellipticity property

$$C_{ijhk}\epsilon_{ij}\epsilon_{hk} \leq c\epsilon_{ij}\epsilon_{hk} \quad \forall\epsilon = \{\epsilon_{ij}\} \in R^3, c \text{ const.} > 0 \tag{2}$$

On Γ_U

$$u_i = U_i, \; U_i = U_i(x, t) \text{ given} \tag{3}$$

and on Γ_F

$$S_i = F_i, \; F_i = F_i(x, t) \text{ given.} \tag{4}$$

The boundary condition on Γ_S reads

$$-\mathbf{S} \in \partial j(u) \text{ on } \Gamma_S. \tag{5}$$

Here j is a convex lower semicontinuous superpotential taking values in the interval $(-\infty, +\infty]$, $j \not\equiv \infty$ (compare with e.g. Refs 14, 15). Note that eqn (5) might be replaced without affecting the method of the present paper by the two conditions

$$-S_N \in \partial j_N(u_N) \quad \text{and} \quad -S_T \in \partial j_T(u_T) \tag{6}$$

The following equations of motion hold:

$$\sigma_{ij,j} + f_i(t) = \rho\ddot{u}_i + c\dot{u}_i \quad \text{in } \Omega \times (0, T) \tag{7}$$

$$\epsilon_{ij} = \epsilon_{ij}(u) = 1/2(u_{i,j} + u_{j,i}) \quad \text{in } \Omega \times (0, T) \tag{8}$$

$$\sigma_{ij} = C_{ijhk}\epsilon_{hk} \quad \text{in } \Omega \times (0, T) \tag{9}$$

$$u_i = u_{i0}(x) \quad \text{at } t = 0 \tag{10a}$$

$$\dot{u}_i = u_{i1}(x) \quad \text{at } t = 0 \tag{10b}$$

where u_{i0} (with respect to u_{i1}) denotes the initial displacements (with respect to velocities), $(0, T)$ is the time interval in which the motion of the body is observed, $f = \{f_i\}$ represents the volume force vector, the comma denotes the partial derivation, $\ddot{u}_i$ is the acceleration vector and ρ is the mass density. Let us assume that the damping term is proportional to the velocity $\dot{u}_i$ and denote the damping coefficient by $c > 0$. The method of m-step linear difference operators is applied for the time discretization of the problem with respect to time. At time instant $t^{(p)}$ one obtains (see Ref. 6)

$$\sum_{r=0}^{q} \alpha^{(r)} u^{(p-r)} = \Delta t \sum_{r=0}^{q} \beta^{(r)} \dot{u}^{(p-r)} \tag{11a}$$

and

$$\sum_{r=0}^{q} \gamma^{(r)} u^{(p-r)} = \mu\Delta t^2 \sum_{r=0}^{q} \beta^{(r)} \ddot{u}^{(p-r)}, \; p \geq q, \; p > 1 \tag{11b}$$

The coefficients $\alpha^{(r)}$, $\beta^{(r)}$, $\gamma^{(r)}$, and μ depend on the finite-difference scheme chosen. The authors assume that the time step size Δt remains constant. Moreover, in order to have an implicit integration scheme $\beta^{(0)}$ is taken to be nonzero. Thus the relations

$$a^{(0)}u^{(p)} - \Delta t\beta^{(0)}\dot{u}^{(p)} = -\mathbf{U}\boldsymbol{\alpha} + \Delta t\dot{\mathbf{U}}\boldsymbol{\beta} \tag{11c}$$

$$\begin{aligned} \mu(\Delta t)^2\beta^{(0)}\ddot{u}^{(p)} &= \gamma^{(0)}u^{(p)} + \mathbf{U}\boldsymbol{\gamma} - \mu(\Delta t)^2\ddot{\mathbf{U}}\boldsymbol{\beta} \\ &= \gamma^{(0)}u^{(p)} + \tilde{\mathbf{U}} \end{aligned} \tag{11d}$$

where

$$\boldsymbol{\alpha} = [\alpha^{(1)}, \ldots, \alpha^{(q)}]^T, \quad \boldsymbol{\beta} = [\beta^{(1)}, \ldots, \beta^{(q)}]^T$$

$$\boldsymbol{\gamma} = [\gamma^{(1)}, \ldots, \gamma^{(q)}]^T, \text{ and } \mathbf{U} = [u^{(p-1)}, \ldots, u^{(p-q)}]$$

are obtained. Accordingly one may write within the p-time interval $(t^{(p)}, t^{(p)} + \Delta t)$ after applying the above time discretization, to the equations of motion, that (omitting index p)

$$\sigma_{ij,j} + f_i(t) = g_i(t) + Au_i \quad A > 0 \text{ in } \Omega \times (t,\ t + \Delta t)$$

Here A is a constant equal to $\gamma^{(0)}\rho/\mu(\Delta t)^2\beta^{(0)} + \alpha^{(0)}c/\Delta t\beta^{(0)}$ which is positive due to the time discretization scheme. The term g_i contains the terms of the previous time steps resulting from eqns (11c) and (11d). Note that thermal terms, and terms due to dislocations, may be included in the model provided they are known at time t.

The authors denote, by $\tilde{f}_i$, the term

$$\tilde{f}_i = f_i - g_i$$

and consider from now on only the behaviour of the structure in the time interval $(t,\ t + \Delta t)$. The above discretization process replaces the dynamic variational inequality formulation of the problem by a static variational formulation. In the absence of shocks it can be shown that as $\Delta t \to 0$ the solution of the dynamic problem is obtained (see Refs 4 and 14). However, in the realistic case of shocks, treated in the present paper, the mathematical problem of convergence is still open.

Let V be the set

$$V = \{v | v = \{v_i\},\ v_i = U_i\ i = 1,\ 2,\ 3 \text{ on } \Gamma_U\} \tag{12}$$

of the kinematically admissible set for the displacements within the time interval $(t,\ t + \Delta t)$. The authors denote the work of the force $f = \{f_i\}$ by $(f,\ v)$ for the displacement $v = \{v_i\}$ on Ω and by $[\bar{f},\ v]_{\bar{\Gamma}}$ the corresponding work on $\bar{\Gamma} \subset \Gamma$, (i.e. $\int_\Omega \bar{f}_i v_i \, d\Omega$, etc.). Further let

$$(u,\ v) = A \int_\Omega u_i v_i \, d\Omega \quad A > 0 \tag{13}$$

and

$$\alpha(u,\ v) = (C\epsilon(u),\ \epsilon(v)) = \int_\Omega C_{ijhk}\epsilon_{ij}(u)\epsilon_{hk}(v) \, d\Omega \tag{14}$$

be the bilinear form of elasticity and let Π be the potential energy within the interval $(t,\ t + \Delta t)$

$$\Pi(v) = 1/2\alpha(v,\ v) + 1/2(v,\ v) - (\tilde{f},\ v) + \Phi(v) - [F,\ v]_{\Gamma_F} \tag{15}$$

where $\Phi(v) = \{\int_{\Gamma_s} j(v) \, d\Gamma$ if $j(v)$ is integrable, ∞ otherwise$\}$.

From Ref. 14 it is known that

$$\Pi(u) = \{\Pi(v) | v \in V\} \text{ in } (t,\ t + \Delta t) \tag{16}$$

characterizes the position of equilibrium within each time interval. Problem (16) has one, and only one, solution (for $v_i \in H^1(\Omega)$-the Sobolev space)

$$C_{ijhk} \in L^\infty(\Omega),\ \tilde{f}_i \in L^2(\Omega),$$

$$F_i \in L^2(\Gamma_F),\ U_i \in H^{1/2}(\Gamma_U)$$

which satisifies equivalently the variational inequality

$$u \in V, \alpha(u,\ v - u) + (u,\ v - u) + \Phi(v) - \Phi(u) - l(v - u) \geq 0 \quad \forall v \in V \tag{17}$$

with $l(v) = (\tilde{f}, v) + [F, v]_{\Gamma_F}$. Relations (16) and (17) are the primal formulations of the BVP within the time interval Δt. For the mixed formulation the authors introduce the translation

$$\bar{u} = u - u_0, \quad \bar{v} = v - u_0 \tag{18}$$

where u_0 is a kinematically admissible displacement field, i.e. such that $u_{0i} = U_i$ on Γ_U and let

$$\bar{u}, \bar{v} \in V_0 = \{v | v = \{v_i\}, v_i \in \tilde{V}, v_i = 0 \quad \text{on} \quad \Gamma_U\} \tag{19}$$

where $\tilde{V} = [H^1(\Omega)]^3$. Then eqn (17) takes the form: find $u = \bar{u} + u_0 \in V$ such that

$$\alpha(\bar{u}, \bar{v} - \bar{u}) + (\bar{u}, \bar{v} - \bar{u}) - l(\bar{v} - \bar{u}) + \Phi(\bar{v} + u_0) - \Phi(\bar{u} + u_0) + a(u_0, \bar{v} - \bar{u}) + (u_0, \bar{v} - \bar{u}) \geq 0 \tag{20}$$

$$\forall v = \bar{v} + u_0 \in V$$

and eqn (16) becomes

$$\bar{\Pi}(\bar{u}) = \min\{\bar{\Pi}(\bar{v}) | \bar{v} \in V_0\} \tag{21}$$

where

$$\bar{\Pi}(\bar{v}) = 1/2(\bar{v}, \bar{v}) + 1/2(\bar{v}, v) - l(\bar{v}) + a(u_0, \bar{v}) + (u_0, \bar{v}) + \Phi(\bar{v} + u_0) = \Pi(\bar{v}) - \Pi(u_0) \tag{22}$$

Note that u_0 may represent any other initial strain field (temperature distributions, given dislocations, etc.). The functional $\bar{l}$ is now denoted by

$$\bar{l}(\bar{v}) = l(\bar{v}) - a(u_0, \bar{v}) - (u_0, \bar{v}) \tag{23}$$

Now let $\mu = \{\mu_i\}$ be the Lagrange multiplier for the problem. Through this Lagrange multiplier the authors introduce the boundary condition (5) on Γ_S.

Let also L be the admissible vector space for μ (see Ref. 15, page 150) and observe that

$$\Phi(\bar{v} + u_0) = \sup_{\mu \in L}([-\mu, \bar{v} + u_0])_{\Gamma_S} - \Phi^c(-\mu) \tag{24}$$

where Φ^c denotes the conjugate function to Φ (see Refs 14 and 15) which is also convex, lower semicontinuous on L and takes values in $(-\infty, +\infty]$, $\Phi^c \not\equiv \infty$. Thus one may write that

$$\inf_{\bar{v} \in V_0} \Pi(\bar{v}) = \inf_{\bar{v} \in V_0} \sup_{\mu \in L} \{\bar{\Pi}(\bar{v}) - [\mu, \bar{v} + u_0]_{\Gamma_S} - \Phi^c(-\mu)\} \tag{25}$$

Thus the Lagrangian of the problem is a real-valued function $\mathfrak{L}$ on $V_0 \times L$ defined within the time interval $(t, t+\Delta t)$ by the relation

$$\mathfrak{L}(\bar{v},\mu) = 1/2a(\bar{v},\bar{v}) + 1/2(\bar{v},\bar{v}) - [\mu, \bar{v}+u_0]_{\Gamma_S} - \Phi^c(-\mu) - [F,\bar{v}]_{\Gamma_F} - (\tilde{f},\bar{v}) + a(u_o,\bar{v}) + (u_0,\bar{v}) \tag{26}$$

and the mixed variational formulation of the semidiscretized variational inequality (17) now reads: find the saddle point

$$\{\bar{w},\lambda\} \in V_0 \times L \quad \text{of} \quad \mathfrak{L} \quad \text{on} \quad V_0 \times L, \text{ i.e.}$$

$$\mathfrak{L}(\bar{w},\mu) \leq \mathfrak{L}(\bar{w},\lambda) \leq \mathfrak{L}(\bar{v},\lambda) \forall \bar{v} \in V_0, \mu \in L \quad \text{in} \quad (t, t+\Delta t) \tag{27}$$

In the previously mentioned functional framework it can be shown, by the methods given in Ekeland & Temam,[5] that problem (27) admits a unique solution $\{\bar{w},\lambda\}$, $\in V_0 \times L$ within the time interval $(t, t+\Delta t)$ such that $\bar{w} = \bar{u} \in V_0$ and $\lambda = S(\bar{u})$ on Γ_S.

3 DERIVATION OF A TIME-DIFFERENCE MULTIVALUED BOUNDARY INTEGRAL EQUATION

Let us consider again the saddle point formulation (27) which is now written in the form: find $\{\bar{w},\lambda\} \in V_0 \times L$ such as to satisfy the problem

$$\mathfrak{L}(\bar{w},\lambda) = \underset{V_0\,L}{\text{infsup}}(\bar{v},\mu) = \underset{L\,V_0}{\text{supinf}}\, \mathfrak{L}(\bar{v},\mu) \text{ in } (t, t+\Delta t) \tag{28}$$

The authors set

$$\inf_{V_0} \mathfrak{L}(\bar{v},\mu) = \tilde{\Pi}_1(\mu) \tag{29}$$

assuming for the present that $\mu \in L$ is given. Then eqn (29) is equivalent to the following classical elasticity problem: find $\bar{u} = \bar{u}(\mu) \in V_0$ such that

$$a(\bar{u},\bar{v}) + (\bar{u},\bar{v}) - [\mu,\bar{v}]_{\Gamma_S} - (\tilde{f},\bar{v}) - (F,\bar{v})_{\Gamma_F} + a(u_0,\bar{v}) + (u_0,\bar{v}) = 0 \quad \forall \bar{v} \in V_0 \text{ in } (t, t+\Delta t) \tag{30}$$

Relation (30) is the expression of the principle of virtual work for a fictive structure, resulting from the initial unilateral one, by eliminating the constraints on Γ_S and by adding the corresponding reactions μ. The position of equilibrium of this fictive structure is characterized within the interval $(t, t+\Delta t)$ by the minimum problem (29) of the potential energy. Owing to the linearity of eqn (30), its solution $\bar{u}$ can be considered as the sum of $\bar{u}_1 \in V_0$ and $\bar{u}_2 \in V_0$ which are the solutions of the two following bilateral problems:

$$a(\bar{u}_1,\bar{v}) + (\bar{u}_1,\bar{v}) - (\tilde{f},\bar{v}) - [F,\bar{v}]_{\Gamma_F} + a(u_0,\bar{v}) + (u_0,\bar{v}) = 0 \quad \forall \bar{v} \in V_0 \tag{31}$$

and

$$a(\bar{u}_2,\bar{v}) + (\bar{u}_2,\bar{v}) - [\mu,\bar{v}]_{\Gamma_S} = 0 \quad \forall v \in V_0 \tag{32}$$

respectively. Both problems describe the equilibrium configuration within each time step of two classical linear elastic bodies resulting from the initial one by considering, at each point, the body's springs with constants $A > 0$, by ignoring the Γ_S support and assuming that the appropriate boundary parts have zero loading. In the case of eqn (31) the structure is loaded by the forces f in Ω and F on Γ_F, whereas on Γ_S the loading is zero. Moreover the initial displacement field u_0 is taken into account. In the case of eqn (32) the structure is subjected to boundary forces $\mu = \{\mu_i\}$, $i = 1, 2, 3$ on Γ_S and the authors assume zero forces in Ω and on Γ_F. Accordingly the solutions $\bar{u}_1$ and $\bar{u}_2$ are uniquely determined, as it is well known from the classical linear elasticity (see Ref. 17). For the arising two classical elasticity problems the solution can be written in terms of Green's operator G. This operator has the same form for both structures, due to the same type of boundary conditions holding in both cases (see Fig. 1). Thus in both cases the solution of the problem can be written in the form

$$\bar{u}_1 = G(\tilde{l}), \bar{u}_2 = G(\mu), \bar{u} = \bar{u}_1 + \bar{u}_2, \tilde{l} = \{\tilde{f}, F, u_0\}, \quad \text{in} \quad (t, t+\Delta t) \tag{33}$$

Note that the yet unknown force distribution $\mu = \{\mu_i\} \in L$ must be determined. From eqns (29), (31) and (32), one obtains, by setting $\bar{v} = \bar{u}_1$ in eqn (31) and $\bar{v} = \bar{u}_2$ in eqn (32)

$$\begin{aligned}
\tilde{\Pi}_1(\mu) &= \mathfrak{L}(\bar{u},\mu) = 1/2a(\bar{u},\bar{u}) - [\mu, \bar{u}+u_0]_{\Gamma_S} \\
&\quad - \Phi^c(-\mu) - (\tilde{f},\bar{u}) - [F,\bar{u}]_{\Gamma_F} + a(u_o,\bar{u}) \\
&= -[\mu,\bar{u}_1]_{\Gamma_S} - 1/2[\mu,\bar{u}_2]_{\Gamma_S} - 1/2(\tilde{f},\bar{u}_1) \\
&\quad - 1/2[F,\bar{u}_1]_{\Gamma_F} - [\mu,u_0]_{\Gamma_S} + 1/2a(u_0,\bar{u}_1) \\
&\quad - \Phi^c(-\mu) \\
&= -[\mu,[G(l)]]_{\Gamma_S} - 1/2[\mu,[G(\mu)]]_{\Gamma_S} \\
&\quad + 1/2a(u_0,G(\tilde{l})) - 1/2(\tilde{f},G(\tilde{l})) \\
&\quad - 1/2(F,G(\tilde{l}))_{\Gamma_F} - \Phi^c(-\mu) - [\mu,u_0]_{\Gamma_S}
\end{aligned} \tag{34}$$

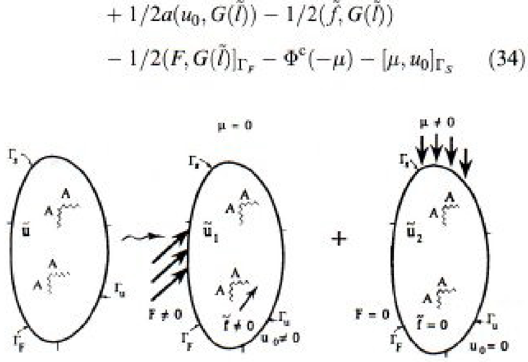

Fig. 1. The problem decomposition (force method) within each step.

where

$$\tilde{a}(u,v) = a(u,v) + (u,v) \tag{35}$$

Further, $\tilde{\beta}$ is denoted by the bilinear form

$$\tilde{\beta}(\mu,v) = [\mu,[G(v)]]_{\Gamma_S} \tag{36}$$

which is symmetric (due to Betti's theorem) and by $\tilde{\gamma}$ the linear form

$$\tilde{\gamma}(\mu) = -[\mu,[G(\tilde{l})]]_{\Gamma_S} - [\mu,u_0]_{\Gamma_S} \tag{37}$$

Thus

$$\begin{aligned}\tilde{\Pi}_1(\mu) &= -1/2\tilde{\beta}(\mu,\mu) + \tilde{\gamma}(\mu) - 1/2(\tilde{f},G(\tilde{l}))\\ &\quad - 1/2[F,G(\tilde{l})]_{\Gamma_F} + 1/2a(u_0,G(\tilde{l}))\\ &\quad + 1/2(u_0,G(\tilde{l}))\end{aligned} \tag{38}$$

From eqns (28), (29) and (38) one can obtain the following minimization problem with respect to the unknown boundary tractions μ:

$$\min\{\Pi_1(\mu) = 1/2\tilde{\beta}(\mu,\mu) + \Phi^c(-\mu) - \tilde{\gamma}(\mu) | \mu \in L\} \tag{39}$$

The term $-1/2(\tilde{f},G(\tilde{l})) - 1/2[F,G(\tilde{l})]_{\Gamma_F} + 1/2a(u_0,G(\tilde{l}))$ does not depend on μ and therefore can be omitted. The authors denote, obviously again, by λ the solution of this problem.

As in Panagiotopoulos[15] it can be shown that $\lambda = S_N(\bar{u})$ on Γ_S, where $\bar{u} \in V_0$ is the solution of the problem (21). Moreover in the functional framework introduced previously in the interval $(t, t+\Delta t)$ one can show that problem (39) has a unique solution. It is worth noting that eqn (39) is equivalent (see Ref. 5) to the variational inequality: find $\lambda \in L$ such as to satisfy

$$\Phi^c(-\mu) - \Phi^c(-\lambda) + \tilde{\beta}(\lambda,\mu-\lambda) - \tilde{\gamma}(\mu-\lambda) \geq 0 \quad \forall \mu \in L \tag{40}$$

Using the definition of the subdifferential, eqn (40) is eqivalent to the 'multivalued' integral equation

$$\gamma - 1/2 \text{ grad } \beta(\lambda,\lambda) \in \partial\Phi^c(-\lambda) \quad \text{on} \quad \Gamma_S \tag{41}$$

which holds on Γ_S.

Applying to eqn (39) the duality theory of convex variational problems (see Refs 5 and 14) a minimum problem with respect to the unknown displacements u and Γ_S is obtained: it reads in the time interval $(t, t+\Delta t)$

$$\min\{\Pi_2(v) - 1/2\tilde{\delta}(v,v) + \Phi(v) - \tilde{\zeta}(v) | v \in V_0\} \tag{42}$$

where $\tilde{\delta}$ and $\tilde{\zeta}$ are obtained from $\tilde{\beta}$ and $\tilde{\gamma}$ respectively by applying the duality transformation of convex analysis (compared with eqn (24)). Note that $\tilde{\delta}$ is a symmetric coercive bilinear form. Notice here two equivalent formulations of eqn (42). The first is: find $u \in V_0$ (compared with eqn (13)) such as to satisfy within $(t, t+\Delta t)$ the variational inequality

$$\Phi(v) - \Phi(u) + \tilde{\delta}(u, v-u) - \tilde{J}(v-u) \geq 0 \quad \forall v \in V_0 \tag{43}$$

The second is: find $u \in V_0$ solution of the multivalued BIE

$$\bar{J} - 1/2 \text{ grad } \tilde{\delta}(u,u) \in \partial\Phi(u) \tag{44}$$

At this point it should be noted that the duality of convex analysis is in the framework of elastomechanics and is equivalent to the duality between the displacement method and force method.

4 DISCRETIZATION AND NUMERICAL STUDY

As it has been pointed out, the solution of the dynamic inequality problems considered in the paper, for each time interval, is formulated after an appropriate time discretization. For the numerical solution, the weighted residual time discretization algorithm proposed by Zienckiewicz, Wood and Taylor,[19] which is a special case of the algorithm defined by eqns (11c) and (11d), has been applied. The algorithm is implicit and unconditionally stable. Note that explicit and conditionally stable algorithms are not applicable because the time step for them is to be chosen on the basis of formulas containing the frequencies of the system, and these formulas do not apply to the present problem; indeed no frequencies can be defined in an inequality problem in the classical sense, as in Ref. 14.

The algorithm independently interpolates the displacement and velocity vectors, and therefore computation of acceleration terms is avoided. This is a significant advantage for the present problem, because of calculation of 'initial' accelerations in the case of impact is avoided.

As the authors have shown in the previous section two minimum problems hold, for each time interval, on the boundary Γ_S of the system which is subjected to inequality boundary conditions. The two minimum problems are dual in the sense that the first eqn (39), has, as unknowns, the boundary forces on Γ_S, whereas the second eqn (42) has, as unknowns, the boundary displacements on Γ_S. In the case of discretized structures, the corresponding discrete analogue of eqns (39) and (42) can be obtained. In this context the authors refer to Ref. 15, with the additional remark that the work of the fictitious springs with constant A must be also considered.

The first minimum problem, eqn (39), as well as the expressions of eqns (31), (32), (36), (37) are considered. From these expressions, which contain energy and mechanical terms, one may obtain an easy method to formulate the discretized minimum problems at the boundary which correspond to eqns (39) and (42). To calculate the discrete form of Π_1 the authors first discretize the elastic bodies under consideration by any discretization method, (eg. FEM or BEM). For this discretized system, the unilateral (inequality) constraints on the boundaries Γ_S refer to the m discrete node pairs of these boundaries. The authors then consider the system Ω_0 obtained from the discretized one by assuming only the kinematical constraints on Γ_U. The resulting structural system is also appropriately modified by the fictitious springs introduced by the time discretization scheme. This discrete system is 'solved' for a pair of unit forces of opposite sign corresponding to an 'inequality' constrained degree of freedom, on the first node pair of Γ_S, and zero forces on the other node pairs of Γ_S. The solution of the resulting underconstrained structure Ω_0, supplies the corresponding displacements in the directions of the 'inequality' constrained degrees of freedom of the m node pairs of Γ_S. They consistute the first column of a matrix **B**. This procedure is repeated for all the node pairs m and thus the whole symmetric positive definite matrix **B** of the influence coefficient is calculated. Note at this point, that if the unit force solutions were analytically given then eqn (41) would be the explicit form of a multivalued integral equation on the boundary.

Within each time interval the displacements in the directions of the 'inequality' degrees of freedom of the node pairs of Γ_S due to the external actions constitute a vector $\tilde{\mathbf{g}}$. Then the discrete form of the minimum problem (39) is written as:

$$\min\{\tfrac{1}{2}\boldsymbol{\mu}^T\tilde{\mathbf{B}}\boldsymbol{\mu} + \Phi^c(-\boldsymbol{\mu}) - \tilde{\mathbf{g}}^T\boldsymbol{\mu} \mid \boldsymbol{\mu} \in L\} \tag{45}$$

Here $\boldsymbol{\mu}$ is the vector of the unknown reactions of Γ_S, which must fulfill the inequality subsidiary conditions of the problem.

The dual discrete minimum problem of eqn (45), which is also the discrete form of eqn (42) can be written as:

$$\min\{\tfrac{1}{2}\mathbf{v}^T\tilde{\mathbf{D}}\mathbf{v} + \Phi(\mathbf{v}) - \tilde{\mathbf{z}}^T \cdot \mathbf{v} \mid \mathbf{v} \in V_0\} \tag{46}$$

where $\mathbf{v}$ is the vector of the unknown displacements on Γ_S, which must fulfill the inequality subsidiary conditions of the problem. By applying the duality relations, between eqns (45) and (46) the matrix $\tilde{\mathbf{D}}$ and the known vector $\tilde{\mathbf{z}}$ are obtained using the elastomechanical duality $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{z}}$ may be obtained also as follows. The discrete structural system Ω is solved by imposing a unit displacement corresponding to an 'inequality' constrained degree of freedom, on the first node of Γ_S by zeroing the displacements of the other degrees of freedom of the same node and of the degrees of freedom of all the other nodes on Γ_S. The solution of the resulting overconstrained structure Ω_0', supplies the corresponding reactions of all the fictitious constraints of the m-nodes of Γ_S. They constitute the first column of a matrix $\tilde{\mathbf{D}}$. This procedure is repeated for all the node

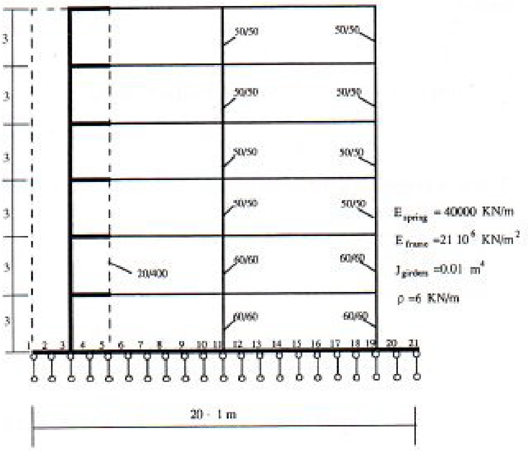

Fig. 2. Analytical model of six-storey frame.

pairs of Γ_S and thus one obtains the whole symmetric positive definite matrix $\tilde{\mathbf{D}}$ of influence coefficients. Within each time interval the reactions of the nodes of all the fictitious constraints of the m nodes of Γ_S due to the external actions constitute the vector $\tilde{\mathbf{z}}$.

Both the discrete minimum problems (45) and (46) are classical quadratic programming problems, symmetric and positive definite, even in the case of elastic bodies (see first example) which do not have any fixed boundary Γ_U $(u_i = 0)$ which prohibit the body's rigid body's motion. This is due to the fact that $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{D}}$ are effective matrices which are calculated from the stiffness matrix of the structure for which fictitious springs have been introduced by the time discretization scheme. $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{D}}$ are full matrices but of small size (only the unknowns on Γ_S are considered).

In the following, two dynamic inequality problems of different nature are numerically studied. In both examples the displacement method is used for the solution. At each time interval $(t, t + \Delta t)$ a modification of Hildreth d'Esopo's algorithm is used by which is obtained the final condition of every pair of nodes on the Γ_S boundaries, through an iterative process. This algorithm is applied for the solution of the quadratic programming problem (46) which has as unknowns only the displacements at the Γ_S boundaries and which converges to the unique solution due to the positive definiteness of matrix $\tilde{\mathbf{D}}$.

As a first example the authors examine the two-dimensional building frame of Fig. 2. The structure is founded on elastic soil through a continuous foundation beam. Unilateral contact conditions hold at the base of the footing since the tensile strength of the soil is negligible. The superstructure is assumed to behave linearly. The masses of the structure are assumed concentrated on the frame girders (6 KN/m) and lumped at the nodal points.

The two dimensional system is subjected to a dynamic

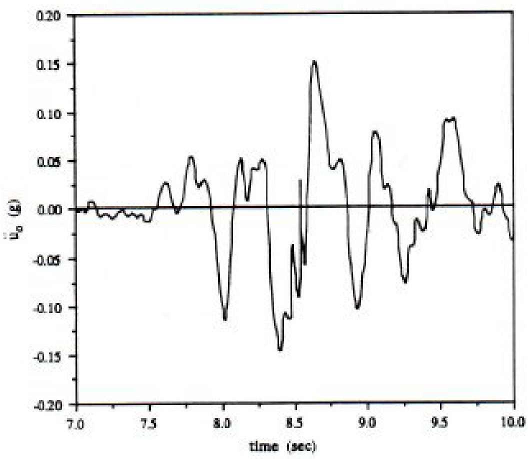

Fig. 3. Strong motion portion of Thessaloniki's 1978 accelerogram.

loading in conjunction with the static loads due to its dead weight. The strong motion of Thessaloniki's 1978 earthquake accelerogram has been selected for the dynamic excitation of the structure (Fig. 3).

The soil is discretized by indpendent linear elastic springs. The unilateral contact law and the loading-unloading path, which is holonomic (path independent) for this problem, are given in Fig. 4. The normal displacements u_N at 21 discrete points of the foundation beam are the unknowns of the minimization problem which is solved at each time interval.

The QP problem (46) which holds for each time interval is written as:

$$\min\{\tfrac{1}{2}\hat{\mathbf{u}}_N^T\bar{D}\hat{\mathbf{u}}_N - \tilde{\mathbf{z}}^T\cdot\hat{\mathbf{u}}_N|\mathbf{u}_N = \hat{\mathbf{u}}_N - \alpha \leq \mathbf{0}\} \qquad (47)$$

where α is a function of already known values of the previous step.

The solution of eqn (47) by the modified Hildreth d'Esopo's algorithm gives the final condition (contact

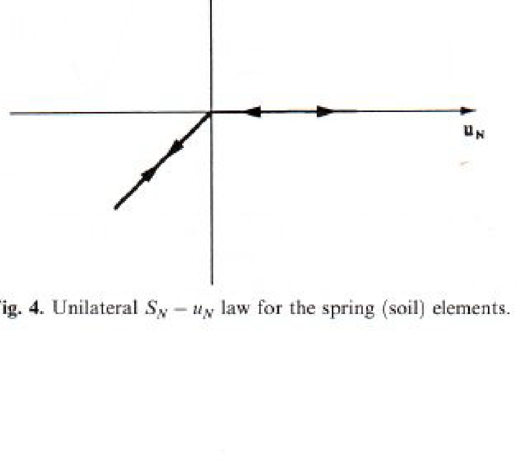

Fig. 4. Unilateral $S_N - u_N$ law for the spring (soil) elements.

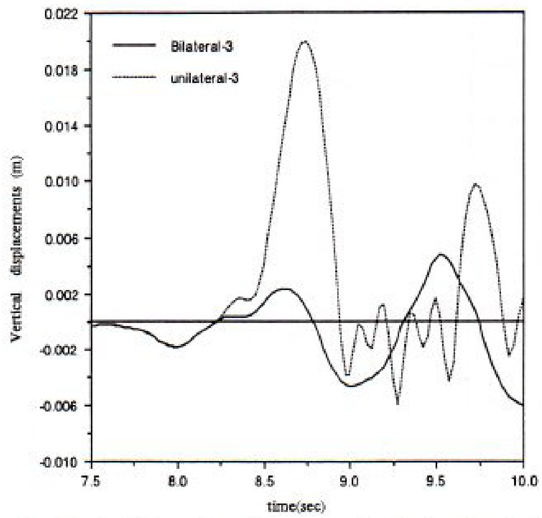

Fig. 5. Oscillations (u_N-displacements) of point 3 of the foundation beam for unilateral and bilateral contact conditions.

or separation) of every node of the foundation beam. Thus, when at a time interval $(t, t+\Delta t)$ a point i of the beam comes into contact with the ground (spring) the algorithm makes the displacements compatible that is $u_N = d_0$ (where d_0 is equal to the gap between the beam node and the spring at the end of the previous time interval $(t-\Delta t, t)$). However, the velocities and accelerations must also be changed due to impact conditions, (Ref. 10). These impact conditions are imposed at this instant at point i. Let $\dot{u}_i^-$ be the vertical velocity of a point i before contact. When the point gets into contact with the ground a part of the kinetic energy is lost. For the examined problem it is reasonable to postulate that all the energy is lost (a perfectly inelastic collision takes place which dissipates the whole kinetic energy). Thus the velocity of the point i just after impact $(\dot{u}_i^+)$ is equal to zero $((\dot{u}_i^+) = 0)$. With the calculated values of u_N and the imposed values of $\dot{\mathbf{u}}_N$, the algorithm proceeds to the next time interval.

The vertical displacements u_N of points 3 and 19 of the continuous footing for unilateral and bilateral contact conditions are presented in Figs 5, 6, 7. Also the normal reaction values for the same points are presented in Figs 8, 9, 10. It is seen in these figures that, for some time, the points of both ends of the continuous beam are in contact. When for some time one end loses contact, generally the other is in contact. Also the reactions and displacements are greatly affected from the bilateral or unilateral assumption of the response of the points of the footing.

As a second example the authors examine a rectangular thin plate in bending with clamped rigid-plastic edges (Fig. 11). The displacements v of any point in

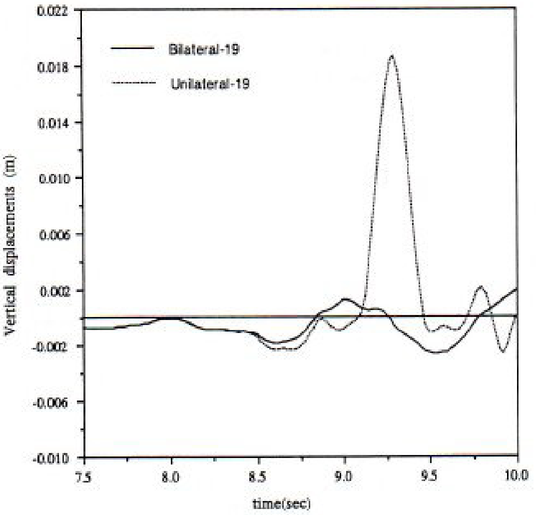

Fig. 6. Oscillations (u_N-displacements) of point 19 of the foundation beam for unilateral and bilateral contact conditions.

the direction of the coordinate axes are expressed through the transverse displacements $w(x, y)$, which are taken as positive upwards.

$$v_i(i = 1, 2, 3) = \left\{ -z\frac{\partial w}{\partial x} = z\theta_y, \quad -z\frac{\partial w}{\partial y} = z\theta_x, w \right\}$$

The plate as shown in Fig. 11 is discretized by four-node rectangular elements. The vector of nodal displacements at any node i is $u_i = \{w_i, \partial w_i/\partial y, -\partial w_i/\partial x\} = \{w_i, \theta_{ix}, \theta_{iy}\}$ and the nodal actions corresponding to them are $p_i = \{p_i, M_{ix}, M_{iy}\}$.

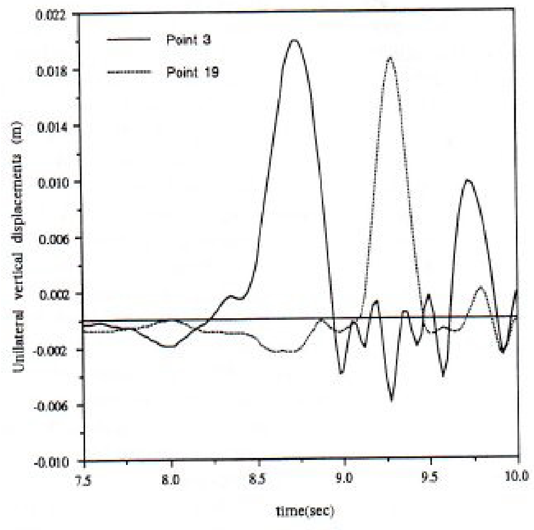

Fig. 7. Displacements of point 3 versus point 19 of the foundation beam for unilateral contact conditions.

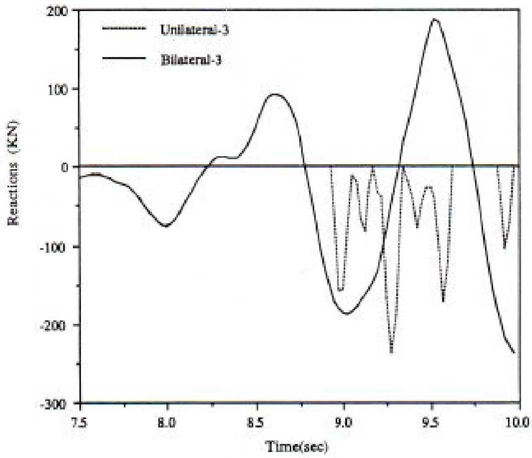

Fig. 8. Reactions at point 3 of the foundation beam for unilateral and bilateral contact conditions.

The response of the plate, under a transverse uniform dynamic load, which varies in time as shown in Fig. 12, is obtained. Plastic hinges may be formed along the boundaries during the loading process. In Fig. 13 is shown the rigid-plastic law which gives the relationship between the edge moments M_T and the inelastic rotations θ_T, at the discrete points of the boundaries, and the loading and unloading paths, which are nonholonomic (path dependent).

M_0^+ and M_0^- are positive values expressing the positive and negative yield moments. For each time interval $(t, t + \Delta t)$ relations between rotations increments $(\Delta\theta_T)$ and the bending moments (M_T) at a point i of the boundary (nonholonomic conditions) must be

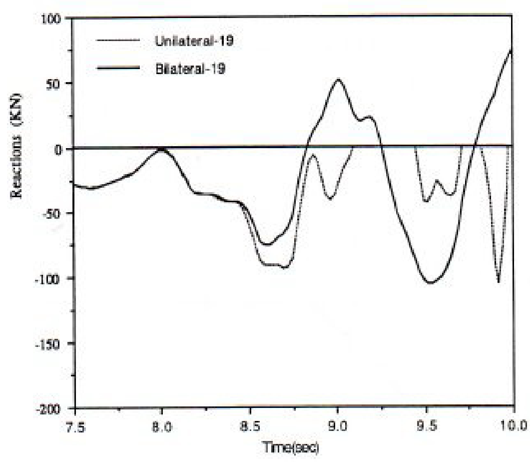

Fig. 9. Reactions at point 19 of the foundation beam and bilateral contact conditions.

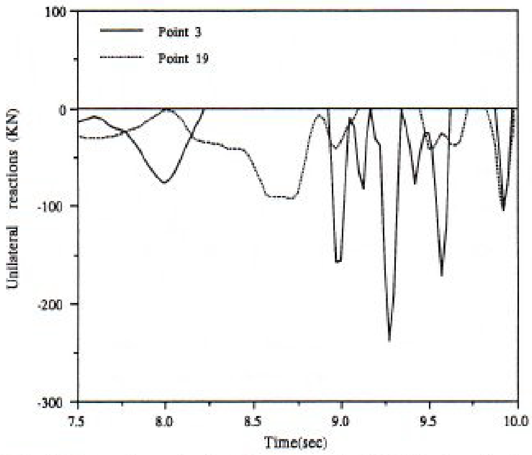

Fig. 10. Reactions of point 3 versus point 19 of the foundation beam for unilateral contact conditions.

written. These relations have the form,[2]

$$\phi_i^+ = M_i - M_{i0}^+ \leq 0, \quad \phi_i^- = -M_i - M_{i0}^- \leq 0 \tag{48}$$

$$\Delta\lambda_i^+ \geq 0, \quad \Delta\lambda_i^- \geq 0 \tag{49}$$

$$\Delta\theta_i = \Delta\lambda_i^+ - \Delta\lambda_i^- \tag{50}$$

$$\phi_i^+ \cdot \Delta\lambda_i^+ = 0, \quad \phi_i^+ \cdot \Delta\lambda_i^- = 0 \tag{51}$$

where the variables ϕ can be interpreted as yield functions. The above relations written in matrix form are:

$$\boldsymbol{\phi}_i = \mathbf{L}_i^T \cdot \mathbf{M}_i - \mathbf{M}_{i0} \quad \text{where} \quad \mathbf{M}_{i0} = \{M_{i0}^+, M_{i0}^-\} \tag{52}$$

$$\Delta\boldsymbol{\lambda}_i \geq 0, \quad \text{where} \quad \Delta\boldsymbol{\lambda}_i = \{\Delta\lambda_i^+, \Delta\lambda_i^-\} \tag{53}$$

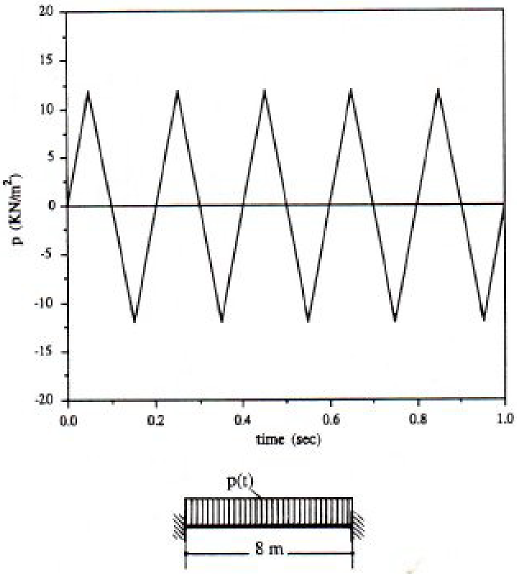

Fig. 12. The dynamic load.

$$\boldsymbol{\phi}_i \cdot \boldsymbol{\lambda}_i = 0 \tag{54}$$

$$\Delta\boldsymbol{\theta}_i = \mathbf{L}_i \cdot \Delta\boldsymbol{\lambda}_i \quad \text{where} \quad \mathbf{L}_i = [1. -1] \tag{55}$$

At first the authors define, as follows, the matrix $\tilde{\mathbf{D}}$ of the convex programming (46). Since the inequality constrained displacements of Γ_S are the rotations θ_T, unit effective rotations $\hat{\theta}_T = 1$, through the technique of Lagrange multipliers, are imposed at the nodes of Γ_S, using the appropriate effective stiffness matrix, (see Ref. 10). At each time interval $(t, t + \Delta t)$ the effective force vector $\tilde{\mathbf{z}}$ (bending moments at the boundary points) is obtained as previously described.

For the total number of constrained boundary nodes,

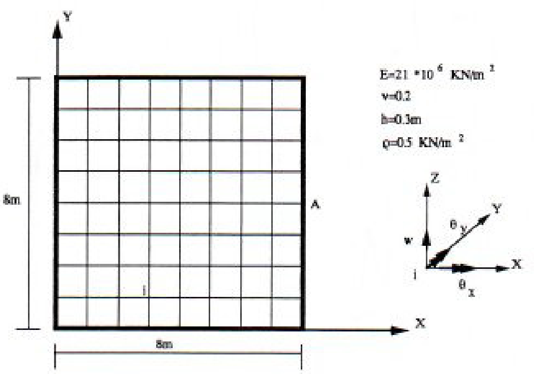

Fig. 11. The geometry and data of a rectangular plate in bending.

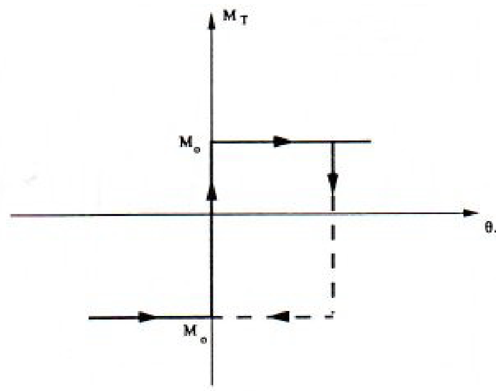

Fig. 13. The rigid-plastic moment-rotation relationship for the boundary of the plate.

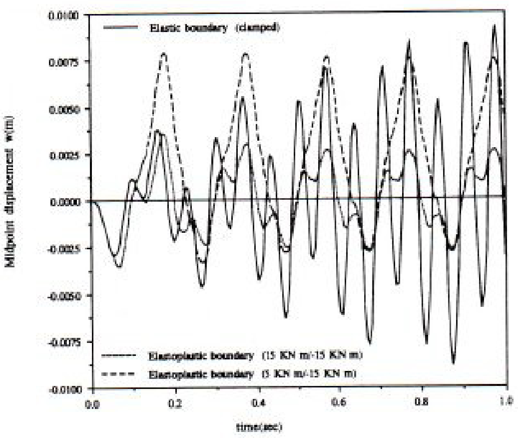

Fig. 14. Oscillations (w-displacements) at the middle point of the plate for three different boundary conditions.

one can write the matrix expressions (52) ÷ (55) as:

$$\boldsymbol{\phi} \leq \mathbf{0} \quad \Delta\boldsymbol{\lambda} \geq \mathbf{0} \quad \boldsymbol{\phi} \cdot \Delta\boldsymbol{\lambda} = 0 \qquad \text{(56a, b, c)}$$

where

$$\boldsymbol{\phi} = [\phi_1, \phi_2, \ldots, \phi_n]$$

$$\Delta\boldsymbol{\lambda} = \{\Delta\lambda_1, \Delta\lambda_2, \ldots, \Delta\lambda_n\}$$

$$\Delta\boldsymbol{\theta} = \{\Delta\theta_1, \Delta\theta_2, \ldots, \Delta\theta_n\} = \mathbf{L} \cdot \Delta\boldsymbol{\lambda}$$

where $\mathbf{L}$ is a block-diagonal supermatrix having $\mathbf{L}_i$ as the main diagonal entries and zero elsewhere.[2] Thus the relations (46) take the form:

$$\min\{\tfrac{1}{2}\Delta\hat{\boldsymbol{\lambda}}^T(\mathbf{L}^T \cdot \bar{\mathbf{D}} \cdot \mathbf{L})\Delta\hat{\boldsymbol{\lambda}}) - (\mathbf{M}_0 + \hat{\mathbf{z}})^T \times \Delta\hat{\boldsymbol{\lambda}} | \Delta\hat{\boldsymbol{\lambda}} - \boldsymbol{\alpha} \leq \mathbf{0}\} \qquad (57)$$

where $\boldsymbol{\alpha}$ is a function of already known values of the previous step and $\hat{M}_0$, $\hat{\mathbf{z}}$, are the functions of $\mathbf{M}_0$ and $\tilde{\mathbf{z}}$, (see Mitsopoulou[10]).

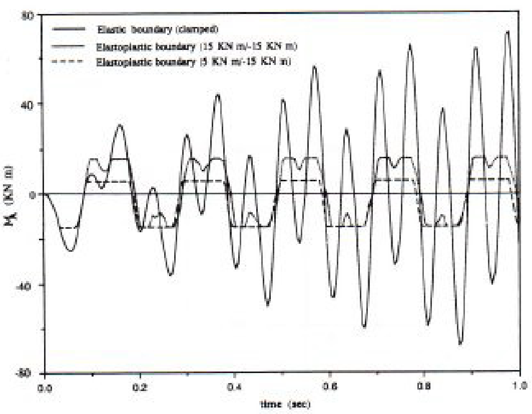

Fig. 15. Bending moments at the middle of the boundary for three boundary conditions.

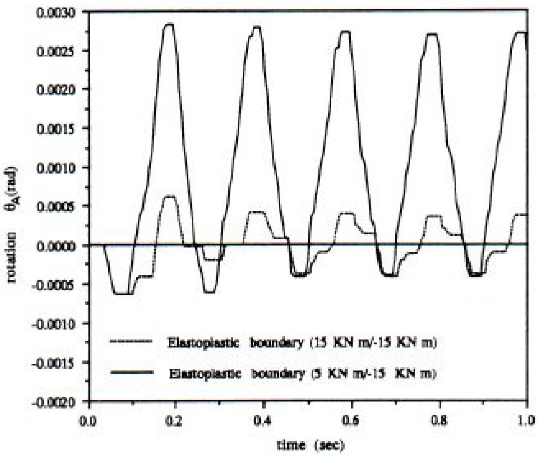

Fig. 16. Rotations at the middle of the boundary for three different boundary conditions.

For the numerical solution of this strictly convex quadratic programming problem, a modification of Hildreth d'Esopo's iterative method is used. At each time interval the algorithm's convergence is very rapid. In Fig. 14 the displacements w of the central node of the plate are shown for three different boundary conditions. In the first case $M_0^+ = M_0^- \to \infty$, in the second $M_0^+ = M_0^- = 16\,\mathrm{KN \cdot m}$ in the third case $M_0^+ = 5\,\mathrm{KN \cdot m}$, $M_0^- = 15\,\mathrm{KN \cdot m}$.

In Fig. 15 the bending moments of the middle point of the boundary are shown. In Fig. 16 the rotations of this point are shown ($\theta_T = 0$ for $M_0^+ = M_0^- \to \infty$). Finally in Fig. 17 the rotations and the bending moments of the midpoint are shown. As it was expected the rotations are changed when the moments take the constant values M_0^+ or M_0^- are conversely when the moments

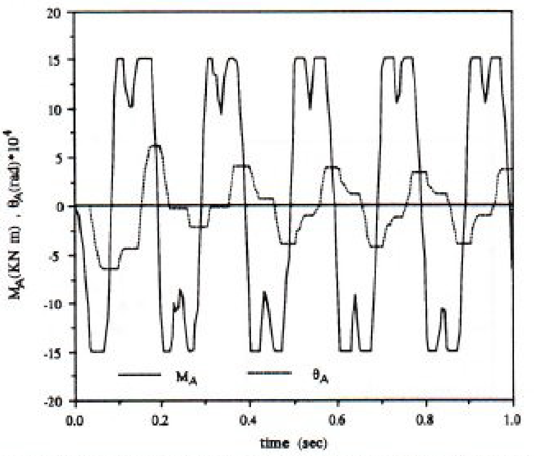

Fig. 17. Bending moments versus rotations at the middle of the boundary for $M_0^+ = M_0^- = 15\,\mathrm{KN \cdot m}$.

are changed, (i.e. they take values between M_0^+ and M_0^-) the rotations remain constant.

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