A NUMERICAL METHOD FOR THE SOLUTION OF THE DYNAMIC UNILATERAL CONTACT-IMPACT PROBLEM

by

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ABSTRACT

In the paper a method is presented for the numerical solution of the dynamic unilateral contact-impact problem between elastic bodies. Spatial finite element discretization, and also temporal discretization are used. For each time step an elimination technique of the internal degrees of freedom gives rise to a minimum quadratic programming (q.p.) problem on the boundary with respect to the unknown unilateral displacements of the time step under consideration. This problem is solved by a relaxation algorithm considering also the impact shocks and the velocities' changes due to them. At the end of the paper numerical examples are presented.

INTRODUCTION

Among the unilateral or inequality problems an important class are the static unilateral contact problems which arise when a deformable body is in "ambiguous" contact with a rigid or a deformable support or with another body with or without friction (Panagiotopoulos[1], Kalker[3], Bisbos[2], Doudoumis[4]). The term "ambiguous" means that we do not know a priory which parts of the bodies are in contact, since tensile stresses cannot be transmitted between the bodies and thus the bodies may separate form one another at some parts of the boundaries.

At the same time another class of problems of considerable importance in science and technology, are the unilateral problems involving dynamic contact - impact effects (Hughes et al[5], Mitsopoulou[6], Ayari[7], Mitsopoulou et al[8]).

In the present paper a method is presented for the numerical treatment of the dynamic unilateral contact impact problem between elastic bodies. Static loads are carried by the structures before the dynamic loading begins. Spatial and temporal discretizations are used and a nonlinear quadratic programming problem is solved at each time step for only a small number of unknowns which are the "unilateral" boundary displacements. Thus the proposed method can be used for the study of the seismic response of large structures supported on soil which is capable of supporting compressive stresses but no tensile stresses (unilateral contact).

FORMULATION OF A DISCRETE MATHEMATICAL MODEL

In order to formulate the discrete mathematical model of the problem mentioned above we assume the following:

a) The displacements and the strains are infinitesimal.
b) The elastic bodies are discretized by proper finite element mesh with n discrete nodes. Thus the configuration of the system is described in the general case, with reference to a global Cartesian coordinate system $Ox_1x_2x_3$, by the $6n$-vector $\boldsymbol{u}$ of generalized nodal displacements (three deflections and three rotations per node).
c) The contact between the bodies is frictionless and is localized at the m nodes of the boundaries Γ_s, for which the unilateral contact conditions hold. The compressive reactions $\boldsymbol{r}_N^\star$ that occur at the contact points are normal (N) to the contact surface ($\boldsymbol{r}_N^\star$ correspond to the unilateral relative displacements $\boldsymbol{u}_N^\star$).

On the basis of the above assumptions and discretization, the dynamic equations of equilibrium of the element assemblage at any time is written as:

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}(t) + \boldsymbol{r}(t) \tag{1}$$

In (1) $\boldsymbol{K}$ is the $6n \times 6n$ stiffness matrix, $\boldsymbol{M}$ is the $6n \times 6n$ positive definite mass matrix and $\boldsymbol{C}$ is the $6n \times 6n$ damping matrix. $\boldsymbol{F}(t)$ is the $6n$ vector of the excitation nidal forces, $\boldsymbol{r}(t)$ the $6n$ vector of the reactions, $\dot{\boldsymbol{u}}$ is the $6n$ vector of the nodal velocities and $\ddot{\boldsymbol{u}}$ the $6n$ vector of the nodal accelerations.

In the paper Rayleigh damping is assumed (Bathe and Wilson[9]) i.e.

$$\boldsymbol{C} = \alpha\boldsymbol{M} + \beta\boldsymbol{K}.$$

Under the hypothesis of continuous displacements and velocities, that is as long as contact-impact does not take place at any boundary point, the motion is fully described by the linear differential equations (1) and by the "initial" conditions at time $t = t_A$:

$$\boldsymbol{u} = \boldsymbol{u}_A \quad \text{and} \quad \dot{\boldsymbol{u}} = \dot{\boldsymbol{u}}_A$$

The "initial" conditions are introduced at time $t = 0$ and at any time step t_A for which discontinuity of the velocities occures due to contact-impact.

The unilateral contact-impact problem will be formulated as a quadratic programming problem after an appropriate time discretization.

Here the implicit and unconditionally stable weighted residual time discretization algorithm proposed by Zienkiewicz, Wood and Taylor[10] is used. The algorithm interpolates independently the displacement and velocity vectors and does not require computation of acceleration terms. This is a significant advantage for the present analysis, because it is avoided the calculation of "initial" accelerations every time t_A at which contact-impact takes place.

On the basis of this algorithm the differential equations of motion (1) are converted to a set of algebraic equations which for the discrete time interval $(t - \Delta t) \div t$ takes the form:

$$\hat{\boldsymbol{K}}\hat{\boldsymbol{U}}_t = \hat{\boldsymbol{F}}_t \tag{2}$$

where

$$\hat{\boldsymbol{K}} = a_1\boldsymbol{K} + a_2\boldsymbol{M} + a_3\boldsymbol{C} \tag{3}$$

$$\hat{\boldsymbol{F}}_t = a_4\boldsymbol{F}_t + a_5\boldsymbol{F}_{t-\Delta t} + \boldsymbol{M}a_8 + \boldsymbol{C}a_9 \tag{4}$$

$$\hat{\boldsymbol{u}}_t = a_6\boldsymbol{u}_t + \boldsymbol{a}_7. \tag{5}$$

Also

$$\dot{\boldsymbol{u}}_t = \big[-(1-\alpha)\dot{\boldsymbol{u}}_{t-\Delta t} + (\hat{\boldsymbol{u}}_t - \boldsymbol{u}_{t-\Delta t})/\theta\cdot\Delta t\big]/\alpha \tag{6}$$

In the previous expressions $a_1 \dots a_6$ are known positive integration constants and $\boldsymbol{a}_7 \dots \boldsymbol{a}_9$ are known vectors.

According to the integration scheme used these constants are:

$$a_1 = \Delta t, \quad a_2 = \frac{1}{\theta\alpha\Delta t}, \quad a_3 = \frac{1}{\alpha}, \quad a_4 = \Delta t\theta$$

$$a_5 = \Delta t(1-\theta), \quad a_6 = \theta,$$

$$\boldsymbol{a}_7 = (1-\theta)\boldsymbol{u}_{t-\Delta t},$$

$$\boldsymbol{a}_8 = -\big(\dot{\boldsymbol{u}}_{t-\Delta t} + \boldsymbol{u}_{t-\Delta t}/\theta\Delta t\big)/\alpha,$$

$$\boldsymbol{a}_9 = -\big((\alpha-\theta)\Delta t\cdot\boldsymbol{u}_{t-\Delta t} - \boldsymbol{u}_{t-\Delta t}\big)/\alpha$$

For α and θ the value 0.5 is used, for which the algorithm is unconditionally stable.

After the above discretization in space and time, at any time interval $(t-\Delta t)\div t$ the discretized problem can take the form of a "static" unilateral contact problem of a fictitious structural system Ω, with (effective) stiffness matrix $\hat{\boldsymbol{K}}$ a load vector $\hat{\boldsymbol{F}}$ and displacements $\hat{\boldsymbol{u}}$. (rel. $3\div 5$). A problem of small size with respect to the unknown relative unilateral displacements $\boldsymbol{u}_N^\star$ $(m \ll 6n)$ is formulated and solved (see also Mitsopoulou, Panagiotopoulos, Zervas[8]).

The solution is obtained as the sum of the effective reactions $\hat{\boldsymbol{r}}_t^0$, $\hat{\boldsymbol{r}}_t^1$, [1]:

$$\hat{\boldsymbol{r}}_t = \hat{\boldsymbol{r}}_t^1 + \hat{\boldsymbol{r}}_t^0 \tag{7}$$

of a **bilateral overconstrained** system Ω', obtained from the system Ω, by assuming that the effective relative displacements on Γ_S are zero. $\hat{\boldsymbol{r}}_t^1$ are the reactions of the system Ω' due to the unknown "strains" $\hat{\boldsymbol{u}}_t$, and $\hat{\boldsymbol{r}}_t^0$ are the reactions due to the external actions when $\hat{\boldsymbol{u}}_t = 0$. Thus relation (7) can be written as:

$$\hat{\boldsymbol{D}}\cdot\hat{\boldsymbol{u}}_t + \hat{\boldsymbol{r}}_t^0 = \hat{\boldsymbol{r}}_t. \tag{8}$$

As a consequence of the above definition the $m\times m$ influence stiffness matrix $\hat{\boldsymbol{D}}$ is determined through calculations at the system Ω' only once (at $t=0$). The influence coefficient $\hat{\boldsymbol{D}}_{ij}$ is defined as the reaction in the direction i due to unit imposed displacement at the direction j. The m-vector $\hat{\boldsymbol{r}}_t^0$ is the reactions of the system Ω' due to the external loads of the increment $((t-\Delta t)\div t)$.

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[1] In the following by $\hat{\boldsymbol{r}}_t$, $\hat{\boldsymbol{u}}_t$, are the normal reactions and displacements of the nodes at the boundary Γ_s at time t.

From rel. (5) we have that

$$\boldsymbol{u}_t = \left(\hat{\boldsymbol{u}}_t - (1-\theta)\cdot \boldsymbol{u}_{t-\Delta t}\right)/\theta \tag{9}$$

Due to the unilateral contact conditions

$$\boldsymbol{u}_t \geq \mathbf{0} \quad \text{or}$$

$$\frac{\hat{\boldsymbol{u}}_t}{\theta} - \frac{1-\theta}{\theta}\boldsymbol{u}_{t-\Delta t} \geq \mathbf{0} \quad \text{or}$$

$$(\hat{\boldsymbol{u}}_t - \boldsymbol{u}_0) \geq \mathbf{0} \quad \text{where} \quad \boldsymbol{u}_0 = (1-\theta)\boldsymbol{u}_{t-\Delta t}$$

For $\hat{\boldsymbol{u}}_t - \boldsymbol{u}_0 = \mathbf{0}$ (that is $\boldsymbol{u}_t = \mathbf{0}$) $\hat{\boldsymbol{r}} > \mathbf{0}$ **(contact condition)**

For $\hat{\boldsymbol{u}}_t - \boldsymbol{u}_0 \geq \mathbf{0}$ $\hat{\boldsymbol{r}} = \mathbf{0}$ **(separation)**

The above relations are written in compact form:

$$\left.\begin{array}{l} \hat{\boldsymbol{u}}_t - \boldsymbol{u}_0 \geq \mathbf{0} \quad \hat{\boldsymbol{r}} \geq \mathbf{0} \\ (\hat{\boldsymbol{u}}_t - \boldsymbol{u}_0)\cdot \hat{\boldsymbol{r}} = \mathbf{0} \end{array}\right\} \tag{10}$$

Relations (10) together with equations (8) form a linear complementarity problem with symmetric positive definite matrix $\hat{\boldsymbol{D}}$ which is equivalent to the following quadratic programming problem:

$$\left\{ \min \frac{1}{2}\hat{\boldsymbol{u}}_t^T \hat{\boldsymbol{D}} \hat{\boldsymbol{u}}_t + \hat{\boldsymbol{r}}_t^{0T} \cdot \hat{\boldsymbol{u}}_t \mid \hat{\boldsymbol{u}}_t - \boldsymbol{u}_0 \geq \mathbf{0} \right\} \quad ^{2}$$

The problem is solved by the Hildreth d'Esopo algorithm and thus the final condition of each nodal pair of points is obtained. At this time if a pair of points i come into contact the velocities change due to impact and proper "initial" conditions must be imposed at pair i (Mitsopoulou[6], Hughes et al[5]). For example if a point i of an elastic body gets into contact with a rigid obstacle the velocity of the point i after impact is equal to zero ($\dot{u}_i^+ = 0$) in case of perfectly inelastic impact, and it is reversed ($\dot{u}_i^+ = -\dot{u}_i^-$) in case of perfectly elastic impact. Using then the already calculated values of u_t and $\dot{u}_t$ on the boundaries together with the load vector at time step $t - \Delta t \div t$, we obtain the displacements and stresses of the system, and we proceed to the next time step.

NUMERICAL EXAMPLES

As a first example a simple uniaxial spring mass system, with mass $m = 1.0$ and stiffness $k = 1.0$, is presented (fig. 1) in which the mass is initially displaced by $x_0 = 1.0$. When perfectly elastic impact is considered the mass oscillates between x_0 and $x_l = -0.5$ where a rigid surface is present, shown in figure 2. It oscillates freely between $-x_l$ and x_l in the case of perfectly inelastic impact (fig. 3).

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[2]The index T means transpose

As a second example the structure shown in fig. 4 is considered. The structure is founded on elastic soil (Winkler assumption) through continuous foundation beams. Unilateral contact conditions hold at the base of the footing.

The system is subjected to a dynamic loading in conjuction with the static loads due to its dead weight. The strong motion of Thessaloniki's (1978) earthquake accelerogram is used for the dynamic excitation of the structure. Perfectly plastic impact is considered. Some characteristic results and comparisons between unilateral and bilateral behaviour are presented in figures (6)÷(8). It is seen that the displacements and the stresses of the structure are greatly affected from the bilateral or unilateral assumption conserning the responce of the foundation.

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[10] Zienkiewicz, O., Wood, W. L., and Taylor, R. L. (1980), An alternative single-step Algorithm for Dynamic Problems, *Earthq. Eng. Struct. Dyn.*, Vol. 8.

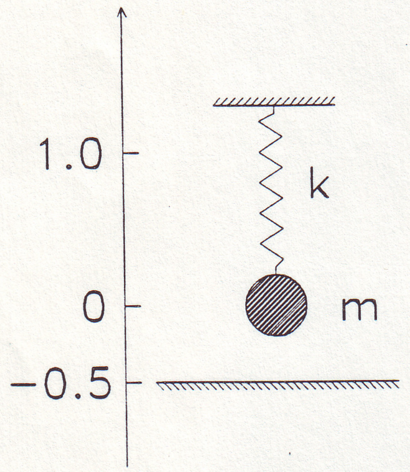

Figure 1: Impact of a spring mass system against a rigid wall.

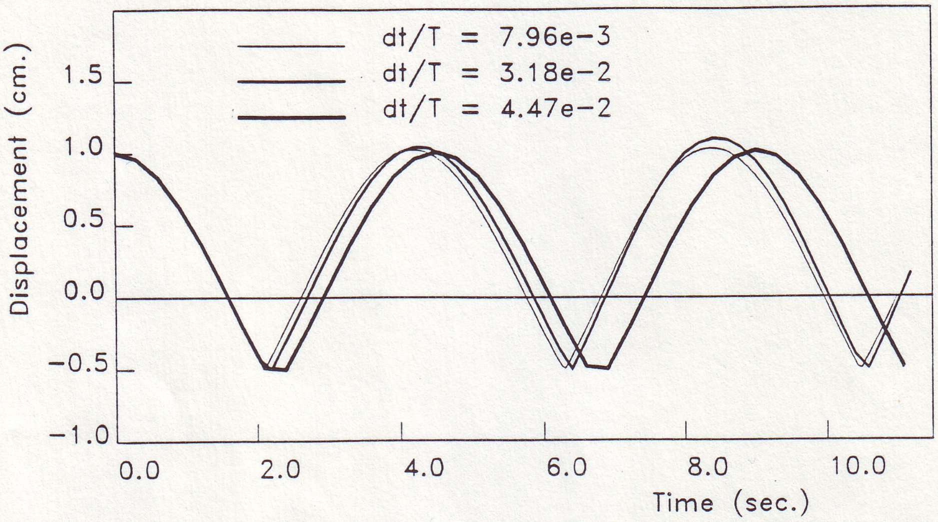

Figure 2: Mass-spring system impact responce for perfectly elastic impact.

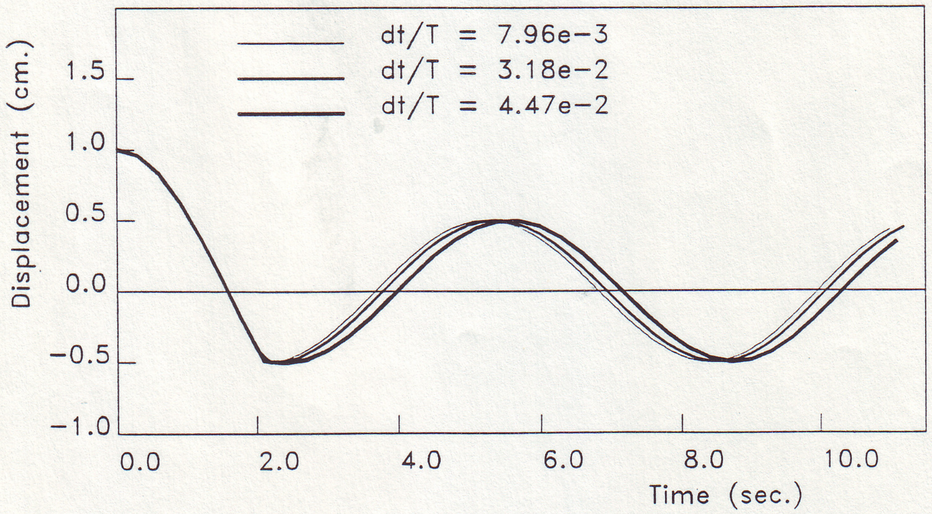

Figure 3: Mass-spring system impact responce for perfectly plastic impact.

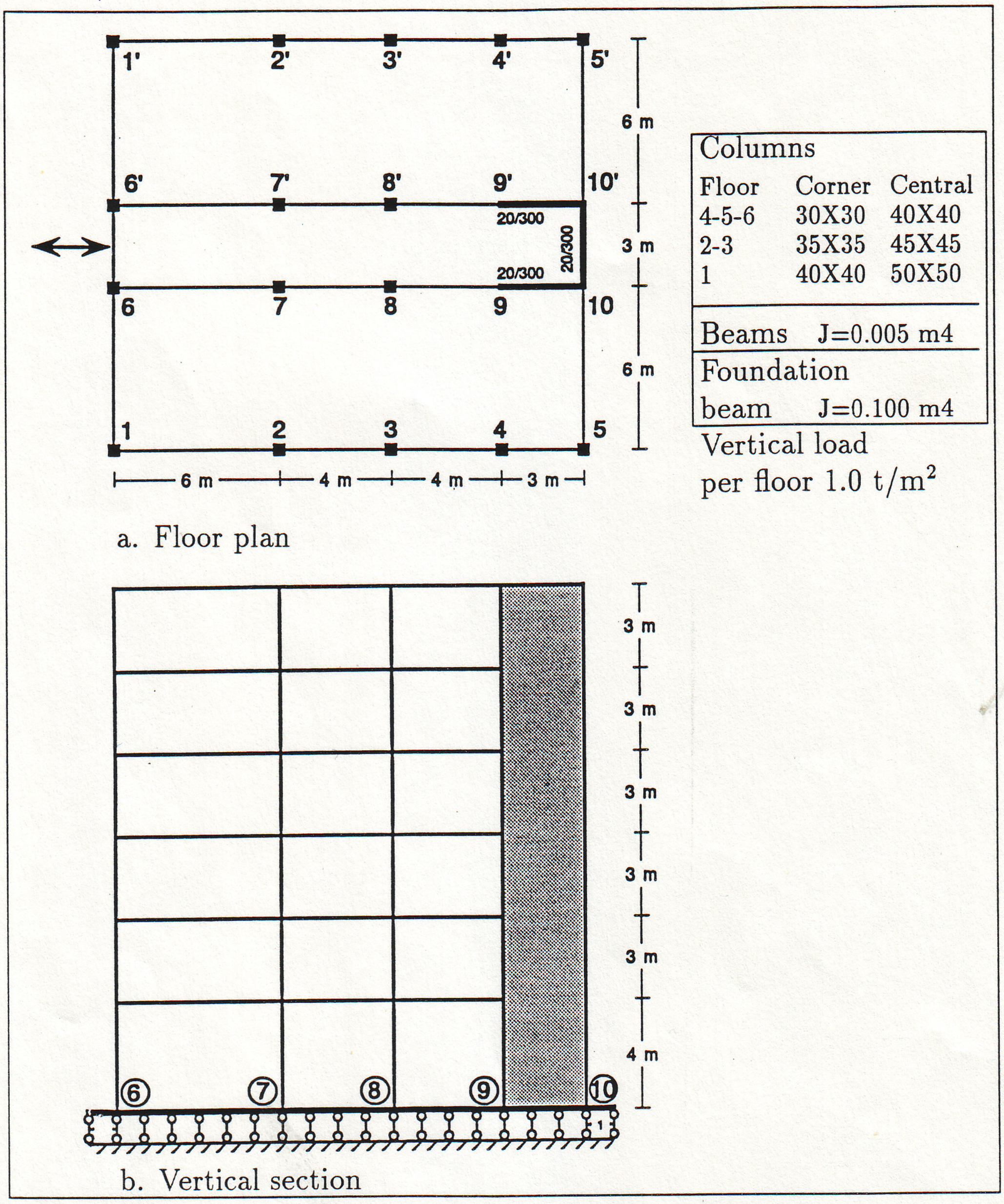

Figure 4: The geometry and data of the examined structure.

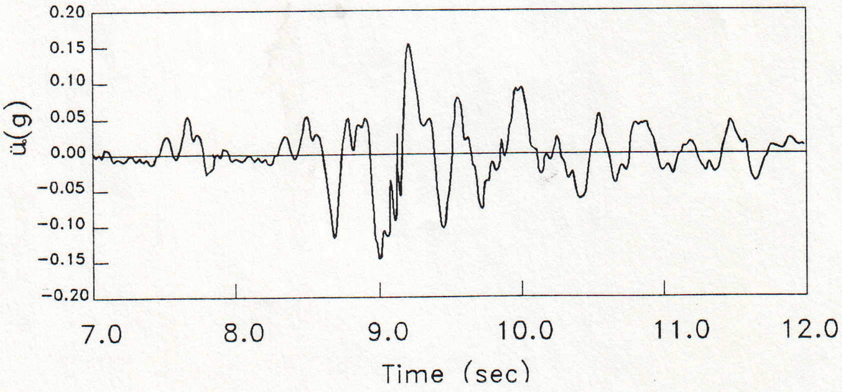

Figure 5: The ground accelerations

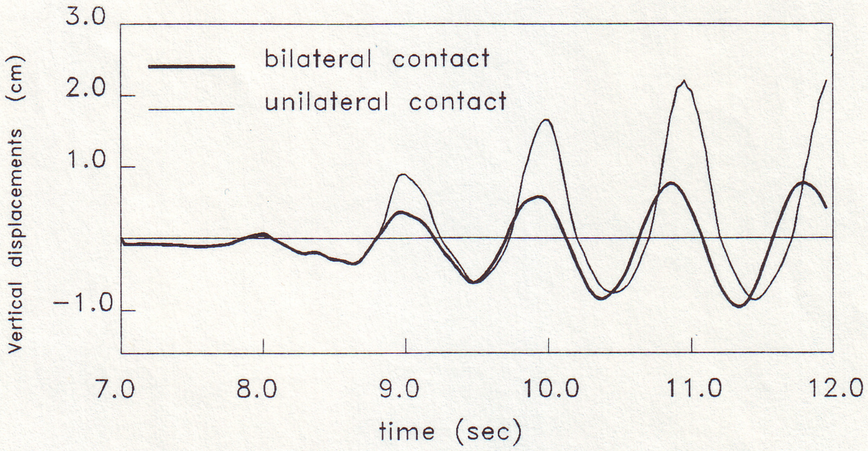

Figure 6: Oscilations (vertical displacements) of the point 10 of the foundation beam for bilateral and unilateral contact soil structure conditions.

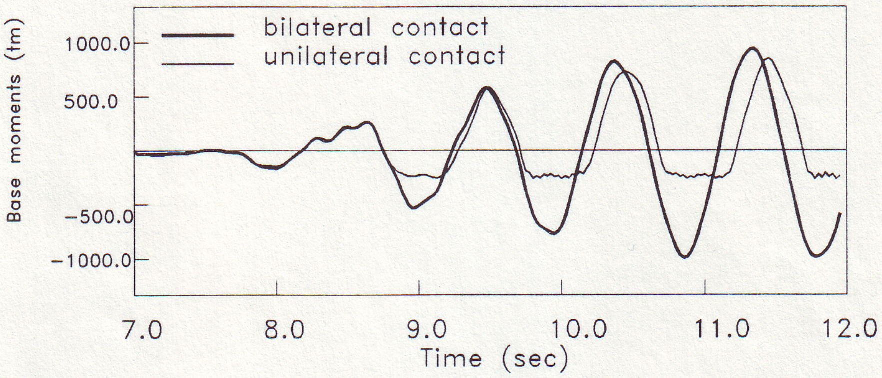

Figure 7: Base moments of the shear wall for bilateral and unilateral contact soil structure conditions.

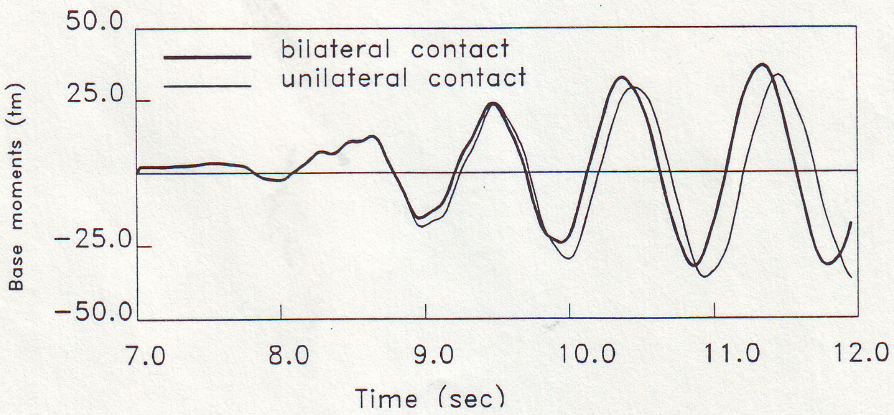

Figure 8: Base moments of column 6 for bilateral and unilateral contact soil structure conditions.